ON FINITE INDEX SUBGROUPS OF A UNIVERSAL GROUP

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ABSTRACT. The orbifold group of the Borromean rings with singular angle 90 degrees, U, is a universal group, because every closed oriented 3–manifold M^3 occurs as a quotient space $M^3 = H^3/G$, where G is a finite index subgroup of U. Therefore, an interesting, but quite difficult problem, is to classify the finite index subgroups of the universal group U. One of the purposes of this paper is to begin this classification. In particular we analyze the classification of the finite index subgroups of U that are generated by rotations.

1. INTRODUCTION

A finite covolume discrete group of isometries of hyperbolic 3–space, H^3, is said to be *universal* if every closed oriented 3–manifold M^3 occurs as a quotient space $M^3 = H^3/G$, where G is a finite index subgroup of the universal group. It was originally shown in [4] that U, the orbifold group of the Borromean rings with singular angle 90 degrees is universal. (See [2] for a simpler proof.)

Although there appear to be infinite families of universal groups, the group U is the only one so far known that is associated to a tessellation of H^3 by regular hyperbolic polyhedra in that there is a tessellation of H^3 by regular dodecahedra with dihedral angles 90 degrees any one of which is a fundamental domain for U.

An interesting, important, but quite difficult problem, is to classify the finite index subgroups of U. A theorem of Armstrong [1] shows that $\pi_1(M^3) \cong G/TOR(G)$ where $TOR(G)$ is the subgroup of G generated by rotations. In particular M^3 is simply connected if and only if G is generated by rotations. One of the purposes of this paper is to begin the classification of the finite index subgroups of U that are generated by rotations. Our main result is Theorem 7.

Theorem 7 *For any integer n there is an index n subgroup of U generated by rotations.*

In Theorem 8 we illustrate the essential differences between the cases n is odd and n is even.

The organization of the paper is as follows: In Section 2 we define the group U, a closely related Euclidean crystallographic group $\widehat{U}$, and a homomorphism $\varphi : U \longrightarrow \widehat{U}$. In Section 3 we show there are tessellations of H^3 by regular dodecahedra and E^3 by cubes and we exploit the homomorphism $\varphi : U \longrightarrow \widehat{U}$ to define a branched covering space map $p : H^3 \longrightarrow E^3$ that respects the two tessellations in the sense that the restriction of p to any one dodecahedron of the tessellation of H^3 is a homeomorphism onto a cube of the tessellation of E^3. In Section 4 we prove the *rectangle theorem* and we use it to classify the finite index subgroups of $\widehat{U}$ that are generated by rotations. In the final section we use this classification together with the homomorphism defined in Section 2 to prove the

main theorem of the paper, Theorem 7, and some existence theorems about finite index subgroups of U generated by rotations.

2. Definitions of U, $\widehat{U}$ and the homomorphism $\varphi : U \longrightarrow \widehat{U}$

Let C_0 be the cube in E^3 with vertices $(\pm 1, \pm 1, \pm 1)$. We obtain a tessellation of E^3 by applying compositions of even integer translations in the x, y, and z directions to C_0. In this paper we do not consider any other tessellations of E^3 and we refer to this tessellation as "the" tessellation of E^3. The intersection of C_0 with the positive octant, together with the lines $\widetilde{a} = (t, 0, 1)$, $\widetilde{b} = (1, t, 0)$, and $\widetilde{c} = (0, 1, t)$; $-\infty < t < \infty$, is depicted in Figure 1.
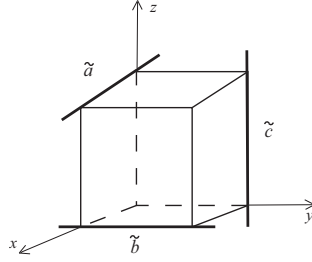

Figure 1

The group $\widehat{U}$ is the Euclidean crystallographic group generated by 180 degree rotations a, b, and c with axes $\widetilde{a}$, $\widetilde{b}$, and $\widetilde{c}$, respectively. We see that $\widehat{U}$ preserves the tessellation and contains the translations $t_x = b(cbc^{-1})$, $t_y = c(aca^{-1})$, $t_z = a(bab^{-1})$, by distances of four, in the x, y, and z directions, respectively.

The cube C_0 is easily seen to be a fundamental domain for $\widehat{U}$, and the axes of rotation in $\widehat{U}$ divide each face of each cube in the tessellation into two rectangles. The quotient space $E^3/\widehat{U}$ is topologically S^3 as can be seen by identifying faces of C_0 using a, b, c and other rotations. The group $\widehat{U}$ is the orbifold group of S^3 as Euclidean orbifold with singular set the Borromean rings B and singular angle 180 degrees. This construction is due to Thurston. For more details see ([6], [2]). The Borromean rings are depicted in Figure 2.
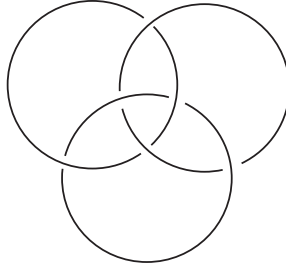

Figure 2. Borromean rings.

The induced map $p : E^3 - \text{preimage } B \longrightarrow (E^3 - \text{preimage } B)/\widehat{U} \approx S^3 - B$ is a regular covering space map so by the theory of covering spaces

$$\widehat{U} \cong \pi_1(S^3 - B)/p_*\pi_1(E^3 - \text{preimage } B).$$

This gives rise to a presentation for $\widehat{U}$:

(1) $\widehat{U} = \langle a, b, c \,|\, a\,b\overline{c}\overline{b}c = b\overline{c}\overline{b}c\,a,\ b\,c\overline{a}\overline{c}a = c\overline{a}\overline{c}a\,b,\ c\,a\overline{b}\overline{a}b = a\overline{b}\overline{a}b\,c,\, a^2,\, b^2,\, c^2 \,\rangle.$

The presentation comes from the usual Wirtinger presentation of the group of the Borromean rings with additional relations a^2, b^2, and c^2 arising from $p_*\pi_1(E^3-$ preimage $B)$ which is normally generated by squares of meridians about the axes $\widetilde{a}$, $\widetilde{b}$, $\widetilde{c}$ of Figure 1.

There is a construction of S^3 as hyperbolic orbifold (also due to Thurston) with singular set the Borromean rings analogous to the previous construction. To describe it we shall work in the Klein model for H^3.

In the Klein model hyperbolic points are Euclidean points inside a ball of radius R centered at the origin in E^3 and hyperbolic lines and planes are the intersections of Euclidean lines and planes with the interior of the ball of radius R. Let D_0 be a regular Euclidean dodecahedron that is symmetric with respect to reflection in the xy, yz, and xz planes. The intersection of D_0 with the positive octant is depicted in Figure 3.
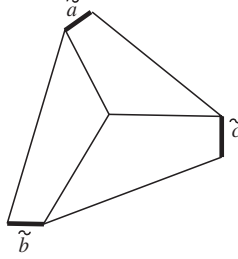

FIGURE 3

If R is chosen correctly, (Details are in [5]), then D_0 can be considered as a regular hyperbolic dodecahedron with 90 degree dihedral angles. Each pentagonal face contains one edge that lies in either the xy, xz, or yz plane. Reflection in this plane, restricted to the pentagon, defines an identification in pairs on the pentagonal faces of D_0. As in the construction with the cube C_0, the resulting topological space is S^3. A hyperbolic orbifold structure is thus induced on S^3 with singular set the Borromean rings, B, and singular angle 90 degrees. The Borromean rings are the image, after identification of the pentagonal edges that lie in the xy, xz, and yz planes.

There is a 4–fold regular branched cyclic covering $q_1 : X^3 \longrightarrow S^3$ with branch set the Borromean rings induced by the natural group homomorphisms

$$\pi_1(S^3 - B) \longrightarrow H_1(S^3 - B; Z) \cong Z \oplus Z \oplus Z \longrightarrow Z \bmod 4.$$

The hyperbolic orbifold structure on S^3 with singular set the Borromean rings pulls back to a hyperbolic manifold (not orbifold) structure on X^3 as meridians are sent to 1 in the above homomorphism.

The hyperbolic manifold X^3 has a *tessellation* consisting of four dodecahedra each of which is sent homeomorphically to D_0 by the map p. The universal covering space map $q_2 : H^3 \longrightarrow X^3$ is used to pull back the tessellation of X^3 by dodecahedra to a tessellation of H^3 by dodecahedra. The composition of covering space maps $q_1 \circ q_2 : H^3 \longrightarrow S^3$ is a regular branched covering space map $H^3 \longrightarrow S^3$ induced by the group of hyperbolic isometries U. That is to say there is a quotient branched covering map $H^3 \longrightarrow H^3/U \approx S^3$ and an associated unbranched covering space map $p : H^3 -$ axes of rotation $= H^3 -$ preimage $B \longrightarrow (H^3 -$ preimage $B)/U \approx S^3 - B$. As in the Euclidean case this covering space map gives rise to a presentation for U via covering space theory:

$$(2) \quad U = \langle a, b, c \,|\, a\, b\overline{c}\overline{b}c = b\overline{c}\overline{b}c\, a,\ b\, c\overline{a}\overline{c}a = c\overline{a}\overline{c}a\, b,\ c\, a\overline{b}\overline{a}b = a\overline{b}\overline{a}b\, c,\ a^4,\ b^4,\ c^4 \rangle$$

As before the presentation comes from the usual Wirtinger presentation of the group of the Borromean rings with additional relations a^4, b^4, c^4 arising from $p_*\pi_1(H^3 - \text{preimage}B)$ which is normally generated by fourth powers of meridians about the axes $\widetilde{a}$, $\widetilde{b}$ and $\widetilde{c}$.

Examining the presentations for U and $\widehat{U}$ we see that they are the same except for the relations a^4, b^4, and c^4 in U and a^2, b^2, and c^2 in $\widehat{U}$. Nonetheless the map $a \to a$, $b \to b$, and $c \to c$, mapping generators of U to generators of $\widehat{U}$, defines a homomorphism $\varphi : U \longrightarrow \widehat{U}$ and an exact sequence.

$$(3) \qquad\qquad 1 \longrightarrow K \longrightarrow U \xrightarrow{\;\varphi\;} \widehat{U} \longrightarrow 1 \,.$$

In this exact sequence K is defined to be the kernel of homomorphism φ.

We say that a group of isometries of H^3 or E^3 is *associated* to a tessellation of H^3 or E^3 by regular compact polyhedra if there is a tessellation of H^3 or E^3 by regular compact polyhedra any one of which is a fundamental domain for the group. Thus the groups U and $\widehat{U}$ are associated to the tessellations of H^3 and E^3 by regular dodecahedra and cubes, respectively. This is not a common occurrence. For example, of the regular polyhedra only cubes can tessellate E^3. In the table below, we have listed the cosines of the dihedral angles of the Euclidean regular polyhedra and also the dihedral angles of the hyperbolic regular polyhedra with vertices on the sphere at infinity. Tetrahedra, octahedra, dodecahedra and icosahedra cannot tessellate E^3 because their dihedral angles are not submultiples of 360 degrees so they don't "fit around an edge".

Polyhedral Type	Euclidean dihedral angle	Hyperbolic dihedral angle vertices at ∞
Tetrahedron	$ArcCos[1/3] \approx 70.5288°$	60°
Cube	$\text{ArcCos}[0] = 90°$	60°
Octahedron	$ArcCos[-1/3] \approx 109.471°$	90°
Dodecahedron	$ArcCos[-1/\sqrt{5}] \approx 116.565°$	60°
Icosahedron	$ArcCos[-\sqrt{5}/3] \approx 138.19°$	108°

There are five regular Euclidean polyhedra but the corresponding hyperbolic polyhedra occur in one parameter families. One can construct the family of hyperbolic cubes, for example, by starting with C_0, the cube with vertices $(\pm 1, \pm 1, \pm 1)$, in the Klein model with the sphere at infinity having Euclidean radius $R = \sqrt{3}$ and let R increase from $\sqrt{3}$ to ∞. There is an isometry from the Klein model using the Euclidean ball of radius R to the Poincaré model using the same Euclidean ball (as Thurston has explained), that is the identity on the sphere at infinity. Since the Poincaré model is conformal and Poincaré hyperbolic planes are Euclidean spheres perpendicular to the sphere at infinity, the dihedral angle between two Poincaré planes is the same as the Euclidean angle between the two circles in which the Poincaré planes intersect the sphere at infinity. Thus the dihedral angle between two Klein planes is the same as the angle between the two circles in which they intersect the sphere at infinity. As R increases, in the case of the cube, for example, from $\sqrt{3}$ to infinity, the dihedral angle increases from 60 degrees to 90 degrees. There exists a compact hyperbolic cube with dihedral angle θ if and only if $0 < \cos\theta < 1/2$. Thus, if it is possible to tessellate H^3 with compact hyperbolic cubes they must have dihedral angle 72 degrees as that is the only submultiple of 360 degrees in the range of possible dihedral angles. A glance at the table (4) indicates that it is impossible to tessellate H^3 with compact regular octahedra or tetrahedra and if it is possible to tessellate H^3 with icosahedra the dihedral angle must be 120 degrees. In the dodecahedral case we have shown that there is a tessellation of H^3 by regular compact hyperbolic dodecahedra with dihedral angle 90

degrees. If there were a different tessellation by compact regular dodecahedra the dihedral angle would have to be 72 degrees.

All the above is part of standard 3–dimensional hyperbolic geometry and we explain it mainly so as to highlight the singular nature of the groups U and $\widehat{U}$ and the tessellations with which they are associated and as background for the following conjecture.

Conjecture *The group U is the only universal group associated to a tessellation of H^3 by regular hyperbolic polyhedra.*

In the next section we study the groups U and $\widehat{U}$ and the tessellations to which they are associated to produce a branched covering of E^3 by H^3.

3. H^3 AS A BRANCHED COVERING OF E^3

Let D_0 and C_0 be the regular dodecahedron and cube in the Klein model for H^3 and in E^3 respectively, as defined in the previous section. We know that D_0 is a fundamental domain for the group U and is also an element of the tessellation of H^3 by regular dodecahedra. For any other dodecahedron D in the tessellation there is a unique element u of U such that $u(D_0) = D$. Analogously, C_0 is a fundamental domain for the group $\widehat{U}$ and is part of the tessellation of E^3 by cubes. For any other cube C in the tessellation there is a unique element $\widehat{u}$ of $\widehat{U}$ such that $\widehat{u}(C_0) = C$.

Let $\alpha_0 : D_0 \longrightarrow C_0$ be a homeomorphism that is as nice as possible. Thus α_0 should commute with reflections in the xy, xz, and yz planes and also with the 3–fold rotations about the axes $\{(t, t, t)\}$ in the Klein model for H^3 and in E^3. The cube C_0 becomes a dodecahedron when each of its faces is split in half by an axis of rotation of $\widehat{U}$. Then α_0, viewed as a map between dodecahedra takes vertices, edges, and faces to vertices, edges, and faces, respectively.

Now we define a map $p : H^3 \longrightarrow E^3$. Let $p = \alpha_0$ on D_0. Any other point A in H^3 belongs to a dodecahedron D of the tessellation. There is a unique $u \in U$ such that $u(D_0) = D$. Let $\widehat{u} = \varphi(u)$ where $\varphi : U \longrightarrow \widehat{U}$ is the homomorphism defined in the previous section. Define the map p by $p(A) = \widehat{u} \circ \alpha_0 \circ u^{-1}(A)$. The map p is well defined for points in the interior of dodecahedra in the tessellation but we must show that p is well defined for the other points. Let A belong to the interior of a pentagonal face P belonging to each of two adjacent dodecahedra D_1 and D_2.

Then there are unique elements u_1 and u_2 of U such that $u_1(D_0) = D_1$ and $u_2(D_0) = D_2$. Then $u_1^{-1}(D_2)$ is a dodecahedron, call it $\widehat{D}$, that intersects D_0 exactly in a pentagonal face P_0. The pentagonal face P_0 of D_0 intersects exactly one of the six axes of rotation, call it ax, that intersect D_0 and this axis lies in the xy, xz, or yz plane of the Klein model. There is a 90 degree rotation about ax, call if rot, that sends D_0 to $\widehat{D}$. Thus $u_1 \circ rot(D_0) = D_2$ which implies $u_1 \circ rot = u_2$, which further implies $\widehat{u_1} \circ \widehat{rot} = \widehat{u_2}$ in group $\widehat{U}$. Then $\widehat{u_2} \circ \alpha_0 \circ u_2^{-1} = \widehat{u_2} \circ \alpha_0 \circ rot^{-1} \circ u_1^{-1} = \widehat{u_1} \circ \widehat{rot} \circ \alpha_0 \circ rot^{-1} \circ u_1$ so that to show that the map p is well defined on the interior of pentagon P it suffices to show that $\widehat{rot} \circ \alpha_0 \circ rot^{-1} = \alpha_0$ when restricted to pentagonal face P_0.

The homomorphism $\varphi : U \longrightarrow \widehat{U}$ takes a, b, and c to $\widehat{a}$, $\widehat{b}$, $\widehat{c}$, respectively where a, b, and c are 90 degree rotations about axes $\widetilde{a}$, $\widetilde{b}$, and $\widetilde{c}$, respectively of Figure 3 and $\widehat{a}$, $\widehat{b}$, $\widehat{c}$ are 180 degree rotations about axes $\widetilde{a}$, $\widetilde{b}$, and $\widetilde{c}$, respectively of Figure 1. The rotation rot is one of a, b, c, a^{-1}, b^{-1}, c^{-1}, bab^{-1}, cbc^{-1}, aca^{-1}, $ba^{-1}b^{-1}$, $cb^{-1}c^{-1}$, $ac^{-1}a^{-1}$. The rotation rot, when restricted to pentagon P_0 equals reflection in the xy, yz, or xz plane depending on which plane axis rot lies in. Similarly, the rotation $\widehat{rot}$ is one of $\widehat{a}$, $\widehat{b}$, $\widehat{c}$, $\widehat{bab^{-1}}$, $\widehat{cbc^{-1}}$, $\widehat{aca^{-1}}$ and the rotation $\widehat{rot}$ when restricted to the half square that is the image of P_0 under α equals reflection in the xy, xz, or yz plane depending on which plane axis $\widehat{rot}$ lies in. But α_0 commutes with reflections

in the xy, xz, or yz planes so that $\widehat{rot} \circ \alpha_0 \circ rot^{-1} = \alpha_0$ and the map p is well defined on the interiors of dodecahedra in the tessellation and on the interiors of their pentagonal faces. That p is also well defined on edges and vertices of the tessellating dodecahedra now follows by a continuity argument.

We summarize all this in a theorem.

Theorem 1. *There exists a tessellation of H^3 by regular hyperbolic dodecahedra with 90 degrees dihedral angle and a tessellation of E^3 by cubes and a map p : $H^3 \longrightarrow E^3$ such that the following holds.*

1. Any dodecahedron in the tessellation of H^3 is a fundamental domain for the universal group U.

2. Any cube in the tessellation of E^3 is a fundamental domain for the Euclidean crystallographic group $\widehat{U}$.

3. The axes of rotation in $\widehat{U}$ divide each face of each cube in the tessellation of E^3 into two rectangles so that the cube may be viewed as a dodecahedron.

4. The restriction of p to any one dodecahedron is a homeomorphism of that dodecahedron onto a cube in the tessellation of E^3. When the cube is viewed as a dodecahedron as in 3 above, the map p sends vertices, edges, and faces to vertices edges and faces respectively. The map p also sends axes of rotation for U homeomorphically, even isometrically, to axes of rotation for $\widehat{U}$.

5. The map p is a branched covering space map with all branching of order two.

In effect, parts 1 through 4 of the theorem have already been proven in the remarks preceding the statement of the theorem. To see that 5 is true, it is only necessary to examine p near an axis of rotation for U. The branching is of order two because four dodecahedra *fit* around every axis of rotation in U while only two cubes *fit* around an axis of rotation of $\widehat{U}$.

It is clear from the definition of the map p when restricted to a dodecahedron, $p = \widehat{u} \circ \alpha_0 \circ u^{-1}$, that the group of covering transformations is the kernel of the homomorphism $\varphi : U \longrightarrow \widehat{U}$. On the other hand p when restricted to (H^3–axes of rotation for U) is an unbranched covering of (E^3 – axes of rotation for $\widehat{U}$) so that $K = \ker \varphi : U \longrightarrow \widehat{U}$ is isomorphic to $\pi_1(E^3$ – axes of rotation for $\widehat{U}$) modulo $p_* \pi_1(H^3$ – axes of rotation for U), by standard covering space theory.

As $\pi_1(E^3$ – axes of rotation for $\widehat{U}$) is a free group generated by meridians, one meridian for each axis of rotation, and $\pi_1(H^3$ – axes of rotation for U) is also generated by meridians it follows that $p_* \pi_1(H^3$ – axes of rotation) is normally generated by squares of meridians, one for each axis of rotation in $\widehat{U}$. We also summarize all this in a theorem.

Theorem 2. *The group of covering transformations for the branched covering $p : H^3 \longrightarrow E^3$ is isomorphic to the group K that is the kernel of $\varphi : U \longrightarrow \widehat{U}$. The group K is naturally isomorphic to a countable free product of Z mod 2's, one generator for each axis of rotation in $\widehat{U}$. In particular the group K is generated by 180 degree rotations.*

As before, the proof of the theorem is in effect given by the remarks immediately prior to the statement of the theorem.

Theorems 1 and 2 enable us to "label" each axis of rotation in U with an algebraic integer in the field $Q(\sqrt{-3})$. Note that each axis of rotation for $\widehat{U}$ is a line of parametric equation $(t, \text{even}, \text{odd})$ or $(\text{odd}, t, \text{even})$ or $(\text{even}, \text{odd}, t)$, $-\infty < t < \infty$. Any such axis intersects the plane $x + y + z = 0$ in a point $(\text{odd}, \text{odd}, \text{even})$ or $(\text{even}, \text{odd}, \text{odd})$ or $(\text{odd}, \text{even}, \text{odd})$ as zero is even. One can verify that the intersection of the tessellation by cubes of E^3 with the plane $x + y + z = 0$ induces a tessellation of the plane $\pi : x + y + z = 0$ by (regular) hexagons and (equilateral) triangles

and that cube C_0 intersects the plane $x + y + z = 0$ in a hexagon with vertices $\{(\pm 1, \mp 1, 0), (\pm 1, 0, \mp 1), (0, \pm 1, \mp 1)\}$. Using a similarity of the plane $x + y + z = 0$ with center the origin and expansion ratio $1/\sqrt{2}$ we can recoordinatize the plane $x + y + z = 0$ by the complex numbers $\mathbb{C}$ so that the six vertices of this hexagon have coordinates equal to the six roots of unity in $\mathbb{C}$. Then every axis of rotation of $\widehat{U}$ intersects the plane $x + y + z = 0$ in a point whose coordinate is an algebraic integer in the field $Q(\sqrt{-3})$. We label each axis d of rotation of U with the coordinate of $p(d) \cap \pi$. Again we summarize these results in a theorem.

Theorem 3. *In the branched covering $p : H^3 \longrightarrow E^3$ each axis of rotation for U is labeled by an algebraic integer of the field $Q(\sqrt{-3})$. The group of covering transformations K preserves labeling. For any two axes of rotation a and b of U with the same label, there is an element k of K such that $k(a) = b$.*

In the next section we classify the subgroups of finite index in $\widehat{U}$ that are generated by rotations.

4. Finite index subgroups of $\widehat{U}$ generated by rotations

The group $\widehat{U}$ is the crystallographic group $I2_1 2_1 2_1$, number 24 of the International Tables of Crystallography [3] . In this section we describe two families of subgroups of $\widehat{U}$ (defined in Section 2) generated by rotations. And we show that *any* finite index subgroup of $\widehat{U}$ generated by rotations is equivalent (in a sense we make precise) to exactly one member of one of the two families.

The axes of rotation of $\widehat{U}$ have parametric equations of form $(t, \text{even}, \text{odd})$, $(\text{odd}, t, \text{even})$ or $(\text{even}, \text{odd}, t)$; $-\infty < t < \infty$ according as to whether they are parallel to the x, y, or z axes. The distance between axes lying in a plane parallel to the xy, xz, or yz planes is an even integer.

Let (m, n, o) be a triple of positive integers where o is odd and m and n are arbitrary. We shall define a group $\widehat{G}(m, n, o)$ associated to the triple (m, n, o) and belonging to the first family by defining a rectangular parallelepiped that will turn out to be a fundamental domain for $\widehat{G}(m, n, o)$.

Let $Box(\widehat{G}(m, n, o))$ be the rectangular parallelepiped defined by the following conditions.

a. The front and back faces of $Box(\widehat{G}(m, n, o))$ lie in the planes $x = 2m + 1$ and $x = -2m + 1$, respectively.

b. The right and left faces of $Box(\widehat{G}(m, n, o))$ lie in the planes $y = 2n$ and $y = -2n$, respectively.

c. The top and bottom of $Box(\widehat{G}(m, n, o))$ lie in the planes $z = o$ and $z = 0$, respectively. $Box(\widehat{G}(m, n, o))$ together with certain axes of rotation is pictured in Figure 4.
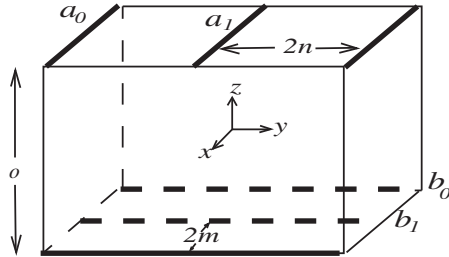

FIGURE 4. $Box(G(m, n, o))$.

Axes a_0, a_1, b_0 and b_1 have parametric equations $(t, -2n, o)$, $(t, 0, o)$, $(-2m + 1, t, 0)$ and $(1, t, 0)$, respectively. Then $\widehat{G}(m, n, o)$ is defined to be the subgroup of $\widehat{U}$ generated by A_0, A_1, B_0, and B_1, the rotations in the axes a_0, a_1, b_0, and b_1, respectively.

Observe that $T_x = B_1 B_0$, $T_y = A_1 A_0$, and $T_z = (A_0 B_1)^2$ are translations by $4m$, $4n$, and $4o$ in the x y, and z directions, respectively. Another generating set of $\widehat{G}(m, n, o)$ is A_1, B_1, T_x, and T_y. Conjugating a translation T_x, T_y, or T_z by a rotation A_1, B_1 either results in the translation itself or its inverse, so there are commutation relations such as $B_1 T_x = T_x^{-1} B_1$. Thus any element of $\widehat{G}(m, n, o)$ has form T, $A_1 T$, $B_1 T$ or $A_1 B_1 T$ where T is a translation that is some product of T_x, T_y, and T_z. With these observations we can see that $Box(\widehat{G}(m, n, o))$ is a fundamental domain for the group $\widehat{G}(m, n, o)$. The volume of $Box(\widehat{G}(m, n, o))$ equals $4m \times 4n \times o$ and the volume of cube C_0, which is a fundamental domain for $\widehat{U}$ equals 8. Thus dividing one by the other, the index of $\widehat{G}(m, n, o)$ in $\widehat{U}$ equals $2mno$, an even integer. The group $\widehat{G}(m, n, o)$ is the crystallographic group $P222_1$, number 17 in the International Tables of Crystallography [3].

Let (p, q, r) be a triple of odd positive integers such that $p \leqq q$ and $p \leqq r$ and if p, q, and r are not all different then $p \leqq q \leqq r$. (The idea here is that any triple of odd positive integers can be cyclicly permuted to a triple satisfying these conditions.) We shall define a group $\widehat{H}(p, q, r)$ in the second family by first defining a rectangular parallelepiped that will turn out to be its fundamental domain. Let $Box(\widehat{H}(p, q, r))$ be the rectangular parallelepiped defined by the following conditions.

The front and back, left and right, top and bottom faces of $Box(\widehat{H}(p, q, r))$ lie in the planes $x = p$, $x = -p$; $y = q$, $y = -q$; $z = r$, $z = -r$, respectively. $Box(\widehat{H}(p, q, r))$ is pictured in Figure 5 along with axes of rotation $a = (t, 0, r)$, $b = (p, t, 0)$, and $c = (0, q, t)$.
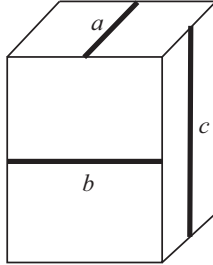

FIGURE 5. $Box(H(p, q, r))$.

The group $\widehat{H}(p, q, r)$ is defined to be the subgroup of $\widehat{U}$ generated by rotations A, B, and C in axes a, b, and c, respectively. Observe that $T_x = BCBC$, $T_y = CACA$, and $T_z = ABAB$ are translations by $2p$, $2q$, and $2r$ in the x, y, and z directions, respectively. Also note that conjugating T_x, T_y, or T_z by (A or B or C) results in T_x or T_x^{-1}, T_y or T_y^{-1}, T_z or T_z^{-1}, respectively. These observations imply that any element of group $\widehat{H}(p, q, r)$ equals exactly one of T, AT, BT or CT where T is a product of T_x, T_y and T_z. As before, we can see that $Box(H(p, q, r))$ is a fundamental domain for group $\widehat{H}(p, q, r)$. The group $\widehat{H}(p, q, r)$ is again the crystallographic group $I2_1 2_1 2_1$, number 24 in [3].

The volume of $Box(H(p, q, r))$ equals $8pqr$ and volume $C_0 = 8$ so, reasoning as before, the index of $\widehat{H}(p, q, r)$ in $\widehat{U}$ is pqr which is an odd integer.

We wish to define an equivalence relation on finite index subgroups of $\widehat{U}$. Let D be the 120 degree rotation about the axis (t, t, t); $-\infty < t < \infty$, which is a main

diagonal of cube C_0 and let $\widehat{S}$ be the group generated by D and $\widehat{U}$. As D has order three and normalizes $\widehat{U}$ we see that $[\widehat{S} : \widehat{U}] = 3$. We define two subgroups of $\widehat{U}$ to be equivalent if they are conjugate as subgroups of $\widehat{S}$. This equivalence relation when applied to the finite index subgroups of $\widehat{U}$ generated by rotations leads to the least messy classification. We shall show that any finite index subgroup of $\widehat{U}$ generated by rotations is equivalent to exactly one $\widehat{G}(m, n, o)$ or $\widehat{H}(p, q, r)$. We observe that rotation D cyclically permutes the x, y, and z axes but that there is no element of $\widehat{S}$ that fixes one of these three axes while interchanging the other two.

For each finite index subgroup $\widehat{G}$ of $\widehat{U}$ we define an "integer triple" (d_1, d_2, d_3). The integer d_1 is the minimal distance between distinct axes of rotation in $\widehat{G}$ that are parallel to the x axis and d_2 and d_3 are similarly defined for rotations about axes parallel to the y and z axis. If $\widehat{G}$ contains no rotations the "integer triple" is (none, none, none). Thus the "integer triple" assigned to $\widehat{G}(m, n, o)$ is (2n, 2m, none) and the integer triple assigned to $\widehat{H}(p, q, r)$ is $(2r, 2p, 2q)$. (Recall, the cubes in the tessellation are $2 \times 2 \times 2$.)

Conjugating a $\widehat{G}(m, n, o)$ or an $\widehat{H}(p, q, r)$ by an element of $\widehat{S}$ at most changes a triple by cyclically permuting it. Thus the fact that $\widehat{G}$ contains no axes of rotation parallel to the z–axis implies that if $\widehat{G}(m, n, o) \sim \widehat{G}(\widetilde{m}, \widetilde{n}, \widetilde{o})$ then $(m, n, o) = (\widetilde{m}, \widetilde{n}, \widetilde{o})$ and the conditions $p \leqq q$ and $p \leqq r$, etc., imply that if $\widehat{H}(p, q, r) \sim \widehat{H}(\widetilde{p}, \widetilde{q}\ \widetilde{r})$ then $(p, q, r) = (\widetilde{p}, \widetilde{q}, \widetilde{r})$. Also as the index of a $\widehat{G}$ in $\widehat{U}$ is even and the index of an $\widehat{H}$ in $\widehat{U}$ is odd no $\widetilde{G}$ can be equivalent to an $\widehat{H}$. The rest of the classification consists of showing that any finite index subgroup of $\widehat{U}$ generated by rotations is either equivalent to an $\widehat{H}(p, q, r)$ or a $\widehat{G}(m, n, o)$.

Suppose that $\widehat{G}$ is a finite index subgroup of $\widehat{U}$ that is generated by rotations. If $\widehat{G}$ contained only rotations parallel to one of the three axes, it would leave planes perpendicular to this axis invariant and thus have infinite index in $\widehat{U}$. So $\widehat{G}$ either contains rotations about axes parallel to two of the three axes x, y, and z or it contains rotations about axes parallel to all three. In the former case, we can assume $\widehat{G}$ contains rotations with axes parallel to the x and y axes but doesn't contain rotations with axes parallel to the z–axis by conjugating by an element of $\widehat{S}$ if need be. In either case let $\mathcal{P}$ be a plane parallel to the yz plane in which an axis of $\widehat{G}$ parallel to the y–axis lies. The set of axes of rotation of $\widehat{G}$ parallel to the x–axis intersects $\mathcal{P}$ in a set of points we call axis points.

Proposition 4. *(The rectangle theorem) There is a tessellation of $\mathcal{P}$ by congruent rectangles with sides parallel to the y and z axes such that the set of axis points equals the set of vertices of the rectangles. Each rectangle is divided in half by an axis of rotation for $\widehat{G}$ parallel to the y–axis.*

The proof of Proposition 4 rests on three facts.

1. If A is a rotation in $\widehat{G}$ with axis ℓ and $S \in \widehat{G}$ then SAS^{-1} is a rotation in $\widehat{G}$ with axis $S(\ell)$. In particular if X is an axis point and $S(\mathcal{P}) = \mathcal{P}$, then $S(X)$ is an axis point.

2. If A is a rotation in $\widehat{G}$ with axis ℓ and T is a translation in $\widehat{G}$ such that $T(\mathcal{P}) = \mathcal{P}$ and $\ell \cap \mathcal{P} = X$ then TA is also a rotation in $\widehat{G}$ and axis $(TA) \cap \mathcal{P}$ is the midpoint of the line segment $XT(X)$.

3. Group $\widehat{G}$ contains translations in the x, y, and z directions. (Because $\widehat{U}$ does and $[\widehat{U} : \widehat{G}] < \infty$.)

Proof of Proposition 4. Let T_y and T_z be translations by minimal distance in the y and z directions respectively, belonging to $\widehat{G}$. (Refer to Figure 6.) Let a_{00} be an axis point. Then by 1 and 2 above, $a_{20} = T_y(a_{00})$ and $a_{10} = $ midpoint $a_{00}a_{20}$ are axis
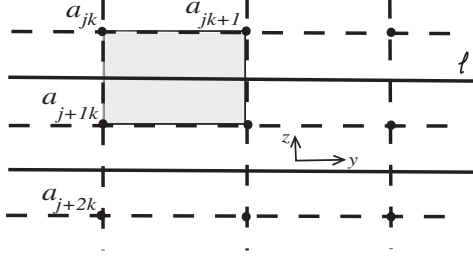
FIGURE 6. The plane $\mathcal{P}$.

points as are $a_{02} = T_z(a_{00})$, $a_{01} = $ midpoint $a_{00}a_{02}$ and $a_{11} = $ midpoint $a_{01}T_y(a_{01})$. The set of vertices of the tessellation by rectangles referred to in Proposition 4 equals $\{T_y^i T_z^j a_{k\ell} \mid i, j \in Z\ k, \ell \in \{0,1\}\}$.

Suppose ℓ is the axis of rotation of B and ℓ lies in plane $\mathcal{P}$, is parallel to the y–axis and intersects the rectangle $R = \{a_{jk}, a_{j+1k}, a_{jk+1}, a_{j+1k+1}\}$, where axis point a_{jk} corresponds to rotation A_{jk}, etc. Then ℓ cannot contain the vertices of R as axes of rotation of distinct elements of $\widehat{U}$ don't intersect. And ℓ must divide R exactly in half for if ℓ lay closer to a_{jk} than to a_{j+1k} the element $A_{jk}(BA_{jk}B^{-1})$ of $\widehat{G}$ would be a translation in the y–direction by a distance less than $a_{jk}a_{jk+2}$ contradicting the minimality in the choice of T_y. The set of translates of the axes ℓ and $A_{jk}(\ell)$ divide every rectangle of the tessellation in half. We must show there are no axis points in $\mathcal{P}$ not of the form a_{jk}. Suppose x was such a point corresponding to rotation X and lying in rectangle $R = \{a_{jk}, a_{jk+1}, a_{j+1k}, a_{j+1k+1}\}$. Then x cannot lie on the sides of the rectangle. (For example, if x lay on $a_{jk}a_{jk+1}$, $XA_{yk}X^{-1}A_{jk}$ would be a translation in the y direction by less than length $a_{jk}a_{jk+2}$ contradicting the minimality in the choice of T_y.) And x cannot lie on ℓ. As x belongs to the interior of the rectangle and not on ℓ, $X(BXB^{-1})$ is a translation in the y–direction by a distance less than $a_{jk}a_{jk+2}$ which is impossible. $\square$

The next problem is to construct a fundamental domain for $\widehat{G}$. With this in mind select a plane $\mathcal{P}$ parallel to the yz plane containing an axis ℓ in $\widehat{G}$ that is parallel to the y–axis. Recall that axes in $\widehat{G}$ parallel to the x, y, or z axis have parametric equations $(t, \text{even}, \text{odd})$, $(\text{odd}, t, \text{even})$ or $(\text{even}, \text{odd}, t)$ respectively. Thus plane P has equation $x = O$ where O is odd. Define the rectangle R_1 in $\mathcal{P}$, as pictured in Figure 7, bounded on one side by ℓ with parametric equation (O, t, e_1) with e_1 even and having the opposite two vertices be axis points for $\mathcal{P}$ with coordinates (O, E, o_1) and $(O, E + 4n, o_1)$ with o_1 odd.
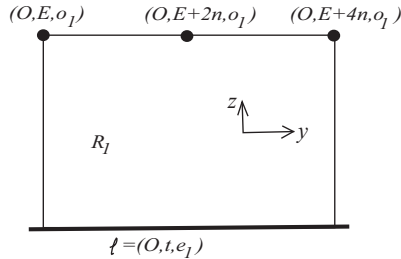

FIGURE 7

There is a rectangle theorem analogous to Proposition 4 but with x substituted for y. Let $\mathcal{Q}$ be the plane $y = E$ which contains the x axis from $\widehat{G}$ with equation

(t, E, o_1). Then $\mathcal{Q}$ also is tesselated by rectangles and we define R_2 to be the rectangle pictured in Figure 8. Like R_1, the rectangle R_2 is not part of the tessellation but is formed by gluing two half–rectangles from the tessellation. R_2 is bounded on one side by axis (t, E, o_1) and the two vertices of R_2 opposite the axis have coordinates (O, E, e_1) and $(O + 4m, E, e_1)$.
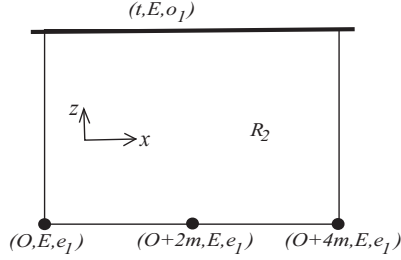

FIGURE 8

Let BOX be that parallelepiped whose projection on planes $\mathcal{P}$ and $\mathcal{Q}$ is rectangles R_1 and R_2, respectively; i.e.,

$$BOX = \{(x, y, z) \mid O \leqq x \leqq O + 4m, E \leqq y \leqq E + 4n, e_1 \leqq z \leqq o_1\}.$$

So the dimensions of BOX are $4m \times 4n \times o$ where $o = e_1 - o_1$ is odd. We assert BOX is a fundamental domain for $\widehat{G}$.

There is a tessellation of E^3 obtained by translating BOX around using translations by $4m$, $4n$, and o in the x, y, and z directions, respectively. One observes, from the rectangle theorems, that the rotations in $\widehat{G}$, which generate $\widehat{G}$, leave this tessellation invariant. Also $\widehat{G}$ contains translations by $4m$, $4n$, and $4o$ in the x, y, and z directions, respectively. Using these translations and the rotations which split the faces of BOX we see that any point in E^3 is equivalent to a point in BOX. If two points in interior of BOX are equivalent then there is a non–trivial element $\widehat{g}$ of $\widehat{G}$ that leaves BOX invariant. By the Brouwer fixed point theorem, $\widehat{g}$ has a fixed point in BOX and therefore must be a rotation whose axis intersects BOX. Inspecting rectangles R_1 and R_2 we see that this is impossible. Thus BOX is a fundamental domain for $\widehat{G}$.

We can conjugate $\widehat{G}$ by an element $\widehat{u}$ of $\widehat{U}$ and obtain an equivalent subgroup of $\widehat{U}$. This has the effect of replacing BOX by $\widehat{u}(BOX)$. As $\widehat{U}$ contains translations by 4 in the x, y, and z directions we may assume without loss of generality that $BOX = \{(x, y, z) \mid \widehat{O} - 2m \leqq x \leqq \widehat{O} + 2m, \widehat{E} - 2n \leqq y \leqq \widehat{E} + 2n, \widehat{e}_1 \leqq z \leqq \widehat{o}_1\}$ where $\widehat{O} = \pm 1$, $\widehat{E} = 0$ or 2, $\widehat{e}_1 = 0$ or 2 and $o = \widehat{o}_1 - \widehat{e}_1$. The rotations $\widehat{a}$, $\widehat{b}$, and $\widehat{c}$ of $\widehat{U}$ are given by equations $(x, y, z) \longrightarrow (x, -y, -z + 2)$, $(x, y, z) \longrightarrow (-x + 2, y, -z)$, $(x, y, z) \longrightarrow (-x, -y + 2, z)$ respectively. So applying $\widehat{a}$, $\widehat{b}$, or $\widehat{c}$ if need be we can assume $\widehat{O} = 1$, $\widehat{E} = 0$ and $\widehat{e}_1 = 0$. But then $BOX = Box(G(m, n, o))$ which implies $\widehat{G} = \widehat{G}(m, n, o)$. We have shown that any finite index subgroup of $\widehat{U}$ generated by rotations that contains rotations with axes in only two of the three possible directions is equivalent to a $\widehat{G}(m, n, o)$.

Now suppose $\widehat{G}$ contains rotations with axes parallel to the x, y, and z directions. For each choice of an ordered pair from the set $\{x$–axis, y–axis, z–axis$\}$ to play the role of y–axis and z–axis in Proposition 4 we get a rectangle theorem. We don't formally state each of the six propositions but we use the results to get tessellations of planes by rectangles in order to construct a parallelepiped, again called BOX, which will turn out to be a fundamental domain for $\widehat{G}$.

Let $\mathcal{P}$ (resp. $\mathcal{Q}$, $\mathcal{R}$) be a plane parallel to the xy (resp. xz, yz) plane containing an axis $a_x = (t, \text{even}, \text{odd})$ (resp. $a_z =(\text{even}, \text{odd}, t)$, $a_y =(\text{odd}, t, \text{even})$) parallel to the x (resp. z, y) axis. Then planes $\mathcal{P}$, $\mathcal{Q}$, and $\mathcal{R}$ intersect in a point $X = (o_1, o_2, o_3)$ with all odd coordinates. (For example, plane $\mathcal{P}$ contains axis $a_x = (t, \text{even}, \text{odd})$ and P is parallel to the xy plane and so has equation $z = \text{odd}$.) No point with all odd coordinates belongs to an axis of rotation in $\widehat{U}$.

Consider the tessellation of plane $\mathcal{P}$ by rectangles. Planes $\mathcal{P}$ and $\mathcal{Q}$ intersect in a line ℓ (see Figure 9) parallel to the x–axis and planes $\mathcal{P}$ and $\mathcal{R}$ intersect in a line m parallel to the y–axis. As $\mathcal{Q}$ contains axes from $\widehat{G}$ parallel to the z–axis line ℓ contains z–axis points that are vertices of the tessellation by rectangles. We already know that the axes in $\mathcal{P}$ parallel to the x–axis evenly divide the rectangles but the line m which is parallel to the y–axis also evenly divides rectangles. To see this translate $\mathcal{P}$ in the z–direction to a plane $\widetilde{\mathcal{P}}$ that contains an axis from $\widehat{G}$ that is parallel to the y–axis. This translation, in the z–direction, takes vertices of the tessellation of $\mathcal{P}$ by rectangles to vertices of the tessellation of $\widetilde{\mathcal{P}}$ by rectangles, leaves plane $\mathcal{R}$ invariant and sends line m to an axis in $\widehat{G}$ parallel to the y–axis that evenly divides a rectangle in $\widetilde{\mathcal{P}}$. Therefore m evenly divides a rectangle of the tessellation of $\mathcal{P}$.

The tessellations of planes $\mathcal{Q}$ and $\mathcal{R}$ by rectangles is also displayed in Figure 9. Planes $\mathcal{Q}$ and $\mathcal{R}$ intersect in line n parallel to the z–axis.
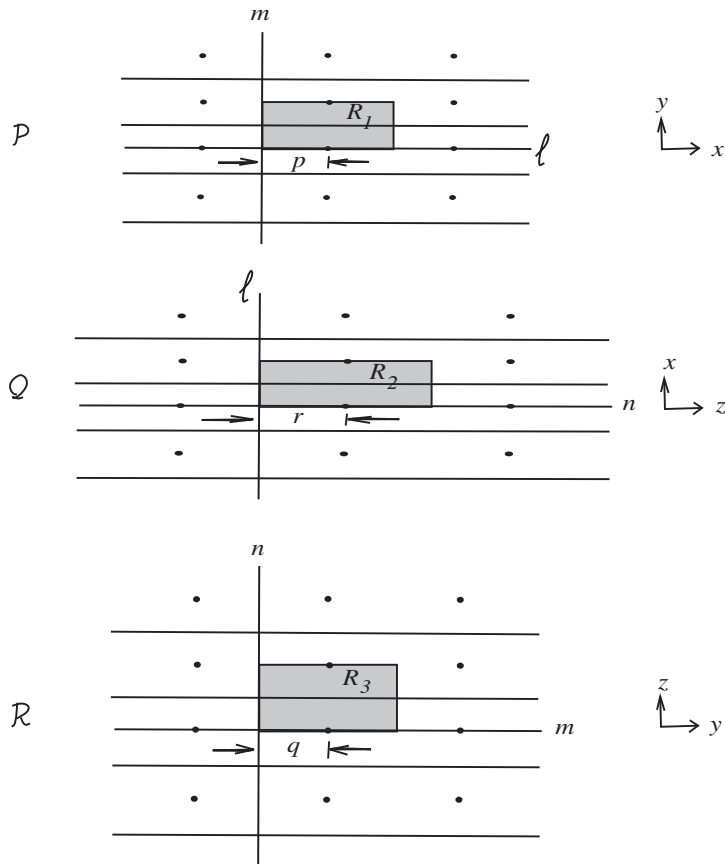

FIGURE 9

The distance from point $X = (o_1, o_2, o_3)$ to the nearest axis in $\widehat{G}$ parallel to the x (resp. y, z) axis is q (resp. r, p) as displayed in Figure 9. That p, q, and r are odd integers can be seen from the parameterizations of the axes in $\widehat{U}$.

Then BOX is defined to be $\{(x,y,z) \mid o_1 \leqq x \leqq o_1 + 2p; o_2 \leqq y \leqq o_2 + 2q; o_3 \leqq z \leqq o_3 + 2r\}$. The projections of BOX on planes $\mathcal{P}$, $\mathcal{Q}$, $\mathcal{R}$ are the rectangles R_1, R_2, R_3 displayed in Figure 9. We assert that BOX is a fundamental domain for $\widehat{G}$. Using translations by $4p$, $4q$, $4r$ in the x, y, and z directions, which are contained in $\widehat{G}$ together with rotations, giving rise to the vertices of the rectangles displayed in Figure 9, we see that any point in E^3 can be moved to a point in BOX. There is a tessellation of E^3 obtained by translating BOX around using translations of $2p$, $2q$, and $2r$ in the x, y, and z directions. From Figure 9 we see that the rotations in $\widehat{G}$ preserve this tessellation so that $\widehat{G}$ preserves this tessellation. If two points in the interior of BOX are equivalent, say $g(m_1) = m_2$, then g preserves BOX, has a fixed point by the Brouwer Theorem and so must be a rotation in $\widehat{G}$ which is impossible.

Therefore BOX is a fundamental domain for $\widehat{G}$. As before we can conjugate by an element of $\widehat{u}$ of $\widehat{U}$ or $\widehat{S}$ which has the effect of replacing $\widehat{G}$ by an equivalent group and BOX by the new fundamental domain $\widehat{u}(\text{BOX})$. Since the center of BOX has all even coordinates $(o_1 + p, o_2 + q, o_3 + r)$ we can find an element $\widehat{u}$ which is a product of translations by 4 in the x, y, and z directions and rotations $\widehat{a}$ and/or $\widehat{b}$ and/or $\widehat{c}$ such that $\widehat{u}(\text{BOX})$ is centered at the origin. Finally we can use the 120 degree rotation D in $\widehat{S}$ to cyclically permute p, q, r so that $p \leqq \max\{q, r\}$ and in the case where p, q, and r are not all different $p \leqq q \leqq r$. Thus the group to which $\widehat{G}$ is equivalent is $\widehat{H}(p, q, r)$ which has $\widehat{u}(\text{BOX})$ as its fundamental domain. We summarize all this as a theorem.

Theorem 5. *1. Let $\widehat{G}$ be an even index subgroup of $\widehat{U}$ generated by rotations. Then $\widehat{G}$ is equivalent to a unique group in the family $\widehat{G}(m, n, o)$ and $\left[\widehat{U} : \widehat{G}\right] = 2mno$. $\widehat{G}$ contains axes of rotation in two of the three directions x, y, and z. The integers $2m$ (resp. $2n$) represents the distance between adjacent axes of $\widehat{G}(m, n, o)$ that lie in a plane parallel to the xy plane and are parallel to the y–axis (resp. x axis). The odd integer o represents this distance between axes of $\widehat{G}$ that are not parallel but are as close as possible.*

2. Let $\widehat{G}$ be an odd index subgroup of $\widehat{U}$ generated by rotations. Then $\widehat{G}$ is equivalent to a unique group in the family $\widehat{H}(p, q, r)$ and $\left[\widehat{U} : \widehat{G}\right] = pqr$. Group $\widehat{G}$ contains rotations with axes parallel to each of the three possible directions x, y, and z.

For each pair of directions x and y, x and z, y and z there is a distance between a pair of axes in these directions that are not parallel but are as close as possible giving rise to a triple of integers. This triple of integers is p, q, and r, not necessarily in that order.

In the next section, we begin the study of finite index subgroups of U that are generated by rotations.

5. Finite index subgroups of U generated by rotations

Proposition 6. *Let $\widehat{G}$ be a finite index subgroup of $\widehat{U}$ generated by rotations and let $G = \varphi^{-1}(\widehat{G})$ be the full preimage of G under the homomorphism $\varphi : U \to \widehat{U}$ defined in Section 2. Then G is generated by rotations.*

Proof. The homomorphism $\varphi : G \to \widehat{G}$ defined in Section 2 is surjective, and sends 90 degree rotations in U to 180 degree rotations in $\widehat{U}$. By the classification of the

$\widehat{G}$ in Section 4, $\widehat{G}$ is generated by 3 or 4 rotations. Let S be a set of 90 degree rotations in U that is sent to a set of generators for $\widehat{G}$ and let G_1 be the subgroup of U generated by S. Then $\varphi^{-1}(\widehat{G}) = G_1 K$. Since K is generated by rotations (Theorem 2), so is $G_1 K$.   □

The main theorem now follows easily from Proposition 6.

Theorem 7. *Given any positive integer n there is a subgroup G of U of index n that is generated by rotations.*

Proof. Let $\widehat{G}$ be a subgroup of $\widehat{U}$ generated by rotations of index n in $\widehat{U}$, which exists by the classification of such subgroups of Section 4. And let $G = \varphi^{-1}(\widehat{G})$. Then G is generated by rotations by Proposition 6 and $[U : G] = [\widehat{U} : \widehat{G}] = n$.   □

Any axis of rotation ℓ in U is the image of the axis of rotation of one of the generators a, b, c of U under the action of an element u of U. This follows from the fact that D_0, a dodecahedral fundamental domain of U, intersects six axes of rotation in U, those of a, b, c, $c^{-1}ac$, $a^{-1}ba$, and $b^{-1}cb$, and if D is any dodecahedron of the tessellation of H^3 intersecting ℓ there is an element u_1 of U such that $u_1(D_0) = D$. Then $u = u_1 x^{-1}$ where x is one of a, b, c. Letting U act on the axes of rotation, we get exactly three orbits. (At most three by the argument above and at least three because $\varphi : U \longrightarrow \widehat{U}$ preserves orbits and there are three orbits in $\widehat{U}$, those parallel to the x, y and z axes.) Thus there are nine conjugacy classes of rotations in U represented by a, a^2, a^3, b, b^2, b^3, c, c^2, and c^3. (This can also be seen by computing $U/[U,U] \cong Z_4 \oplus Z_4 \oplus Z_4$ from the presentation of U in Section 2. For example a is sent to $(1, 0, 0)$, etc.) Similarly there are three conjugacy classes of rotations in $\widehat{U}$ represented by $\widehat{a}$, $\widehat{b}$, and $\widehat{c}$.

Theorem 8. *Let G be a subgroup of U of odd index and generated by rotations. Then G contains a member of each of the nine conjugacy classes of rotations in U.*

Proof. Let $\widehat{G} = \varphi(G)$ where $\varphi : U \longrightarrow \widehat{U}$ is the homomorphism of Section 2 and $K = \ker \varphi$. Then $G \subset GK \subset U$ so that $[U : G] = [U : GK][GK : G]$. But $\varphi : U \longrightarrow \widehat{U}$ induces $\varphi : GK \longrightarrow \widehat{G}$ so that $[U : GK] = [\widehat{U} : \widehat{G}]$ and $[U : G] = [\widehat{U} : \widehat{G}] \cdot [GK : G]$. Since $[U : G]$ is odd it follows that $[\widehat{U} : \widehat{G}]$ is odd and thus $\widehat{G}$ contains a member of each of the three conjugacy classes of rotations in $\widehat{U}$ from the classification in Section 4.

We shall show that G contains a member of the conjugacy class of c. Let $\widehat{c}_1$ be a rotation in $\widehat{G}$ with axis parallel to the z–axis. Suppose $\varphi(g) = \widehat{c}_1$. Then g is a product of rotations, $g = \prod\limits_{i=1}^{n} r_i$, as G is generated by rotations. If $\{r_1, \ldots, r_n\}$ contains a rotation conjugate to c or c^3 we are done. Suppose this is not the case. Then $\widehat{c}_1 = \prod\limits_{i=1}^{n} \widehat{r}_i$ where $\widehat{r}_i$ is either the identity or a rotation about an axis parallel to the x or y axes. Each $\widehat{r}_i$ belongs to the group $\widehat{G}(1, 1, 1)$ defined in Section 4 as that group contains every rotation in $\widehat{U}$ about an axis parallel to the x or y axis. Thus $\widehat{c}_1 \in \widehat{G}(1, 1, 1)$ but this is impossible as $\widehat{G}(1, 1, 1)$ contains no rotations with axis parallel to the z–axis. Therefore G contains a member of the conjugacy class of c.

The two conjugates $D\widehat{G}(1, 1, 1)D^{-1}$ and $D^2\widehat{G}(1, 1, 1)D^{-2}$, where D is 120 degree rotation about axis (t, t, t) introduced in Section 4, contain all rotations parallel to the x and z axis and no rotation parallel to the y axis or all rotations parallel to the y and z axes and no rotation parallel to the x axis. We can show that G contains rotations in the conjugacy class of a and b by duplicating the argument for c by replacing $\widehat{G}(1, 1, 1)$ by $D\widehat{G}(1, 1, 1)D^{-1}$ or $D^2\widehat{G}(1, 1, 1)D^{-2}$.   □

If G is a finite index subgroup of U that is generated by rotations it is clear that information about the precise placement of $\widehat{G}$ in the classification of Section 4 implies much about group G itself. There are other theorems analogous to Theorems 7 and 8, but clumsier to state or prove that we could present. We refrain from doing so, so as not to lengthen this paper.

We close by posing a question. If G is a subgroup of U of index n, either generated by rotations or not, it is clear that G has a fundamental domain that is a union of n of the dodecahedra in the tessellation associated to U. Does G have a fundamental domain that is convex and also the union of n dodecahedra?

References

[1] M.A. Armstrong. The fundamental group of the orbit space of a discontinuous group. *Math. Proc. Cambridge Philos. Soc.*, 64:299–301, 1968.

[2] H.M. Brumfield, G.and Hilden, M. T. Lozano, J. M. Montesinos, E. Ramírez-Losada, H. Short, D. Tejada, and M. toro. Three manifolds as geometric branched coverings of the three sphere. *arXiv:0710.1960v1[math.GT]*, 2007.

[3] Theo Hahn (Editor). *International Tables for Crystallography*. IUCr-Springer, 2005.

[4] H. M. Hilden, M. T. Lozano, J. M. Montesinos, and W. C. Whitten. On universal groups and three-manifolds. *Invent. Math.*, 87(3):441–456, 1987.

[5] H.M. Hilden, M.T. Lozano, and J. M. Montesinos-Amilibia. The arithmetic structure of a universal group. *Atti Sem. Mat. Fis. Univ. Modena*, 49(suppl.):1–14, 2001. Dedicated to the memory of Professor M. Pezzana (Italian).

[6] W. P. Thurston. *Three-dimensional geometry and topology. Vol. 1*, volume 35 of *Princeton Mathematical Series*. Princeton University Press, Princeton, NJ, 1997. Edited by Silvio Levy.

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