On subdirect products of free pro-p groups and Demushkin groups of infinite depth

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Abstract

We study subdirect products of free and Demushkin pro-p groups of depth ∞ developing theory similar to the abstract case, see [4]. Furthermore we classify when a subdirect product has homological type FP_m for some $m \geq 2$, a problem still open for abstract groups for $m \geq 3$.

1 Introduction

In this paper we study homological properties of pro-p groups, in particular the homological type FP_m. A pro-p group G is of type FP_m if there is a projective resolution of the trivial $\mathbb{Z}_p[[G]]$-module $\mathbb{Z}_p$ with all modules finitely generated up to dimension m. Here $\mathbb{Z}_p[[G]]$ is the completed group algebra of G with coefficients in $\mathbb{Z}_p$, the pro-p completion of $\mathbb{Z}$. In the case of abstract groups there is a similar definition with $\mathbb{Z}_p$ replaced by $\mathbb{Z}$ and the completed group algebra replaced by the ordinary group algebra. In the abstract case there is a stronger property F_m that is finite presentability for $m = 2$ and in general F_m is equivalent to FP_m together with finite presentability if $m \geq 2$. In the pro-p setting the difference between these properties disappears i.e. a pro-p group has type FP_2 if and only if it is finitely presented as a pro-p group.

Our main result is a classification of subdirect products of type FP_m of specific pro-p groups. In the abstract case the homological or homotopical properties of subgroups of direct products of free groups were first studied by Stallings

*Partially supported by "bolsa de produtividade de pesquisa" from CNPq, Brazil 2000 AMS Math. Subject Classification 20J05, 20E18 who gave the first example of a finitely presented group that is not of type FP_3. Later, Baumslag and Roseblade [2] showed that a subgroup of type FP_∞ of a direct product of finitely generated free groups that intersects each factor non-trivially and maps surjectively to each factor of the direct product has finite index. It turned out that this holds for surface groups too [4] and recently was proved for abstract limit groups [5].

In this paper we treat only pro-p groups. We show that a pro-p version of the main result of [4] holds where surface groups are replaced by Demushkin groups of infinite depth i.e. pro-p completions of orientable surface groups. Recall that a subgroup H of $D = G_1 \times \ldots \times G_n$ is a subdirect product if for every canonical projection $p_i : D \to G_i$ we have that $p_i(H) = G_i$.

Theorem A. *Let each of $G_1, \ldots, G_n$ be a free non-procyclic pro-p group or a non-abelian Demushkin group of depth $q = \infty$ and let $H \subseteq D = G_1 \times G_2 \times \ldots \times G_n$ be a closed subdirect product i.e. H is a closed subgroup of D that projects surjectively to every G_i. Suppose further that H is finitely presented as a pro-p group and $H \cap G_i \neq 1$ for every $1 \leq i \leq n$. Then H is of type FP_m if and only if for every projection $p_{j_1,\ldots,j_m} : D \to G_{j_1} \times \ldots \times G_{j_m}$ we have that $p_{j_1,\ldots,j_m}(H)$ has finite index in $G_{j_1} \times \ldots \times G_{j_m}$.*

As a corollary we deduce the following result

Corollary B. *Let each of $G_1, \ldots, G_n$ be a free non-procyclic pro-p group or a non-abelian Demushkin group of depth $q = \infty$ and let $H \subseteq D = G_1 \times G_2 \times \ldots \times G_n$ be a closed subdirect product such that H has homological type FP_n and $H \cap G_i \neq 1$ for every $1 \leq i \leq n$. Then $(H \cap G_1) \times (H \cap G_2) \times \ldots \times (H \cap G_n)$ is a subgroup of finite index in H.*

The main obstacle to transferring the result from the abstract case to the pro-p case is that geometric methods are not usually transferrable to the pro-p case. In the case of abstract groups a geometric result due to P. Scott [17] plays an important role in the proof of the results of Section 1.2 of [4] about primitive elements in surface groups. In the pro-p case we prove a similar result (see Theorem 2) using an approximation technique from the proof of the classification of Demushkin groups, see [20, Ch. 12. 3].

In a recent preprint [11] a new class of pro-p groups was defined that shares many properties with abstract limit groups : commutative transitive, finite cohomological dimension, type FP_∞, non-positive Euler characteristic, free-by-nilpotent. The groups in this class were called pro-p limit groups as their definition uses the extension of centralizer approach from one of the definitions of abstract limit groups. The Demushkin groups of infinite depth with $d(G) = d$, the minimal number of generators, divisible by 4 are pro-p limit groups but even in the case $d = 6, p \neq 2$ we do not know whether this is the case. Still it is tempting to conjecture that both Theorem A and Corollary B hold for pro-p limit groups.

Conjecture C. *Both Theorem A and Corollary B hold if $G_1, \ldots, G_n$ are non-abelian pro-p limit groups.*

2 Preliminaries

For a pro-p group

$$G = \varprojlim_i G_i,$$

where G_i are finite p-quotients of G, the completed group algebra with coefficients in k, where $k = \mathbb{Z}_p$ or $k = \mathbb{F}_p$, is a local ring defined by

$$k[[G]] = \varprojlim_i k[[G_i]].$$

A pro-p $k[[G]]$ module M is a pro-p additive group equipped with a continuous G-action i.e. M is an inverse limit of its p-primary finite $k[[G]]$-quotients (i.e. finite quotients of order a power of p). A subset Y of M is a set of topological generators of M if the smallest closed $k[[G]]$-submodule of M that contains Y is M. A subset Y generates M topologically if and only if the image of Y in $M\widehat{\otimes}_{k[[G]]}\mathbb{F}_p$ is a set of topological generators as a pro-p $\mathbb{F}_p$-module, here $\widehat{\otimes}_R$ is the completed tensor product over a profinite ring R. In particular M is (topologically) finitely generated over $k[[G]]$ if and only if $M\widehat{\otimes}_{k[[G]]}\mathbb{F}_p$ is finite (note that since $\mathbb{F}_p$ is finitely generated as a $k[[G]]$-module $M\widehat{\otimes}_{k[[G]]}\mathbb{F}_p \simeq M\otimes_{k[[G]]}\mathbb{F}_p$). Then by the Nakayama lemma, M is abstractly finitely generated as a $k[[G]]$-module if and only if it is topologically finitely generated. By [20, Lemma 7.2.2] for M and N (top.) finitely generated pro-p $k[[G]]$-modules, any abstract $k[[G]]$-homomorphism $\varphi : M \to N$ is continuous, hence a homomorphism of pro-p $k[[G]]$-modules.

Infinite Demushkin groups are pro-p Poincaré duality groups of dimension 2. The classification of such groups was started in [6], [7] and completed in [12], [18]. Such groups have an invariant q associated with them called depth and the simplest case is when $q \neq 2$ (note that the depth is always a power of p or infinity). A Demushkin group of depth $q \neq 2$ has a pro-p presentation $\langle y_1, y_2, \ldots, y_d \mid y_1^q[y_1, y_2] \ldots [y_{d-1}, y_d]\rangle$, where d is even and by definition $y_1^\infty = 1$, a detailed proof can be found in [20, Ch. 12.3]. In the case when $q = \infty$ a Demushkin group is the pro-p completion of the orientable surface group. Pro-p Poincaré duality groups G share some of the properties of the abstract Poincaré duality groups, for example a subgroup of infinite index in G has cohomological dimension strictly smaller than the cohomological dimension of G [14, Ch. iii, 7, Exer. 3b)]. In particular every subgroup of infinite index in G of cohomological dimension 2 is a free pro-p subgroup.

By $Kdim(k[[G]])$ of $k[[G]]$ we denote the Krull dimension of the abstract (non-necessary commutative) rings suggested in [15]. In particular we will consider the Krull dimension of abstract finitely generated $k[[G]]$-modules (remember that topologically finitely generated pro-p $k[[G]]$-modules are abstractly finitely generated $k[[G]]$-modules and vice-versa, the topology is hidden in the topological ring $k[[G]]$). We will be interested only in the case when G is a finite rank pro-p group, in particular a (topologically) finitely generated nilpotent pro-p group. By the main results of [1] for a nilpotent pro-p group G of finite rank $Kdim(\mathbb{Z}_p[[G]]) = Kdim(\mathbb{F}_p[[G]]) + 1 = d + 1$, where d is the pro-p version of Hirsch length of G i.e. the number of copies of $\mathbb{Z}_p$ in any sequence of subnormal subgroups of G with pro-cyclic quotients.

For a pro-p group G we define inductively $\gamma_1(G) = G$ and $\gamma_i(G) = \overline{[\gamma_{i-1}(G), G]}$, where overlining stands for closure.

Some of the proofs of our results use commutator calculations and we fix the basic commutator $[a, b]$ as $a^{-1}b^{-1}ab$ following the notations of [20], note the definition of basic commutator in [4] is slightly different. We denote by a^b the conjugate $b^{-1}ab$.

3 Auxiliary results on Demushkin groups

Proposition 1. *Let G be a Demushkin group of depth $q = \infty$ and N be a non-trivial closed normal subgroup of G. Then there is a subgroup of finite index G_0 in G such that G_0 has a pro-p presentation*

$$\langle z_1, z_2, \ldots, z_d \mid r \rangle,$$

where d is even, $\widetilde{F}$ is the free pro-p group with basis $z_1, z_2, \ldots, z_d$, $\pi : \widetilde{F} \to G_0$ is the canonical projection and

$$r \equiv [z_1, z_2][z_3, z_4] \ldots [z_{d-1}, z_d] \text{ modulo } [[\widetilde{F}, \widetilde{F}], \widetilde{F}] \text{ and } z_1, z_2 \in \pi^{-1}(N \cap G_0).$$

Proof. By going down to a subgroup of finite index in G if necessary we can assume that the image of N in $G/[G, G]G^p$ is non-trivial. By the classification of Demushkin groups, see [20, Ch. 12.3], G has a pro-p presentation

$$\langle x_1, \ldots, x_s \mid [x_1, x_2][x_3, x_4] \ldots [x_{s-1}, x_s] \rangle,$$

s is the minimal number of (topological) generators of G and s is even. Furthermore x_1 can be chosen arbitrary in $F \setminus [F, F]F^p$ modulo $[F, F]$, where F is a free pro-p group with a basis $x_1, \ldots, x_s$ i.e. we can assume that the image of x_1 in G is in $N[G, G]$ [20, Lemma 12.3.7]. Denote by e_i the image of x_i in $V = F/[F, F]$, which is a free $\mathbb{Z}_p$-module with basis $\{e_1, \ldots, e_s\}$.

Consider the isomorphism (with respect to the basis $\{e_1, \ldots, e_s\}$ of V) between $V \wedge V$ and the anti-symmetric $\mathbb{Z}_p$-linear maps $V \times V \to \mathbb{Z}_p$ that sends $\sum_{1 \leq i < j \leq s} z_{ij} e_i \wedge e_j$ to $f : V \times V \to \mathbb{Z}_p$ such that $f(e_i, e_j) = z_{ij}$. We view the image of $[x_1, x_2][x_3, x_4] \ldots [x_{s-1}, x_s]$ in $F/[[F, F], F]$ as an element of $V \wedge V$ and hence as an anti-symmetric bilinear form φ on V i.e.

$$\varphi(e_i, e_j) = 0 \text{ for } \mid j - i \mid \neq 1 \text{ or } i < j, i \text{ even},$$

$$\varphi(e_1, e_2) = \varphi(e_3, e_4) = \ldots = \varphi(e_{s-1}, e_s) = 1.$$

Case 1. For some $v \in V \setminus \mathbb{Z}_p e_1$ in the image of N in V, $\varphi(e_1, v) \in \mathbb{Z}_p \setminus p\mathbb{Z}_p$.

By substituting $\varphi(e_1, v)^{-1}v$ for v, we can assume that $\varphi(e_1, v) = 1$. Then there is a $\mathbb{Z}_p$-basis $\{e_1, v\} \cup \{v_i = e_i + \alpha_i e_1\}_{3 \leq i \leq s}$ of V for some $\alpha_i \in \mathbb{Z}_p$ such that

$e_1 \wedge e_2 + e_3 \wedge e_4 + \ldots + e_{s-1} \wedge e_s = e_1 \wedge v + v_3 \wedge v_4 + \ldots + v_{s-1} \wedge v_s \in V \wedge V$. This basis of V lifts to a basis $y_1 = x_1, y_2, \ldots, y_s$ of F such that

$$r \equiv [y_1, y_2][y_3, y_4] \ldots [y_{s-1}, y_s] \text{ modulo } [[F, F], F]$$

and the images of y_1, y_2 in G are in $N[G, G]$. Then there are elements $\widetilde{y}_1, \widetilde{y}_2$ of F such that $\widetilde{y}_1 y_1^{-1}, \widetilde{y}_2 y_2^{-1} \in [F, F]$ and the images of $\widetilde{y}_1, \widetilde{y}_2$ in G are in N. Note that

$$[y_1, y_2][y_3, y_4] \ldots [y_{s-1}, y_s] \equiv [\widetilde{y}_1, \widetilde{y}_2][y_3, y_4] \ldots [y_{s-1}, y_s] \text{ modulo } [[F, F], F].$$

Thus $\widetilde{y}_1, \widetilde{y}_2, y_3, \ldots, y_s$ is the required basis of F and we are done.

Case 2. For every v in the image of N in V, either $v \in \mathbb{Z}_p e_1$ or $\varphi(e_1, v) \in p\mathbb{Z}_p$. In this case, consider the map $\chi : G \to \mathbb{Z}/p\mathbb{Z}$ that sends x_1 to 1 and $\{x_i\}_{2 \le i \le s}$ to 0 and define G_0 as the kernel of χ. Note that G_0 is (topologically) generated by the images of

$$\widetilde{X} = \{x_1^p, x_i^{x_1^j}\}_{2 \le i \le s, 0 \le j \le p-1} = \{z_1, z_2, \ldots, z_d\}$$

in G, where $d = (s-1)p + 1$ and by the Schreier formula the above set is a basis of a free pro-p subgroup $\widetilde{F}$ in F. Note that for $w = [x_1, x_2] \ldots [x_{s-1}, x_s]$

$$w^{x_1^j} = [x_1, x_2^{x_1^j}][x_3^{x_1^j}, x_4^{x_1^j}] \ldots [x_{s-1}^{x_1^j}, x_s^{x_1^j}] =$$

$$(x_2^{x_1^{j+1}})^{-1} x_2^{x_1^j} [x_3^{x_1^j}, x_4^{x_1^j}] \ldots [x_{s-1}^{x_1^j}, x_s^{x_1^j}].$$

Then

$$\widetilde{w} = \prod_{j=p-1}^{0} w^{x_1^j} \equiv (x_2^{x_1^p})^{-1} x_2 \prod_{j=p-1}^{0} [x_3^{x_1^j}, x_4^{x_1^j}] \ldots [x_{s-1}^{x_1^j}, x_s^{x_1^j}] =$$

$$[x_1^p, x_2] \prod_{j=p-1}^{0} [x_3^{x_1^j}, x_4^{x_1^j}] \ldots [x_{s-1}^{x_1^j}, x_s^{x_1^j}] \text{ modulo } [[\widetilde{F}, \widetilde{F}], \widetilde{F}]$$

is a relation of G_0.

Let μ be the anti-symmetric bilinear form of $V_0 = G_0/[G_0, G_0]$ corresponding to $\widetilde{w}$ with respect to the basis Z_0 of $V_0 = G_0/[G_0, G_0]$ that is the image of $\widetilde{X}$ in V_0. Denote by s_i the image of x_i in G. Note that since N is normal in G and $s_1 \in N[G, G] \subseteq NG_0$

$$s_3^{s_1} s_3^{-1} = [s_1, s_3^{-1}] \in [NG_0, G_0] \subseteq (N[G_0, G_0]) \cap G_0 = (N \cap G_0)[G_0, G_0],$$

and similarly $s_4^{s_1^2} s_4^{-1}, s_4^{s_1} s_4^{-1} \in (N \cap G_0)[G_0, G_0]$, hence

$$s_4^{s_1^2} (s_4^{s_1})^{-1} \in (N \cap G_0)[G_0, G_0].$$

Then for the images v_1 and v_2 of $s_3^{s_1} s_3^{-1}$ and $s_4^{s_1^2} (s_4^{s_1})^{-1}$ in $G_0/[G_0, G_0]$

$$\mu(v_1, v_2) = -1 \in \mathbb{Z}_p \setminus p\mathbb{Z}_p.$$

If ν is the anti-symmetric bilinear form of V_0 (with respect to the basis Z_0) corresponding to the unique relation of G_0 (remember G_0 is a Demushkin group with depth $q = \infty$) then $\mu = r\nu$ for some $r \in \mathbb{Z}_p$. Thus $\nu(v_1, v_2) \in \mathbb{Z}_p \setminus p\mathbb{Z}_p$ and v_1 and v_2 are elements of the image of $N \cap G_0$ in V_0 such that the images of v_1 and v_2 in V_0/pV_0 are linearly independent. Then we can continue as in the first paragraph of the proof. □

Theorem 2. *Let G be a Demushkin group of depth $q = \infty$ and N be a non-trivial closed normal subgroup of G. Then there is a subgroup of finite index G_0 in G such that G_0 has a pro-p presentation*

$$\langle y_1, y_2, \ldots, y_d \mid [y_1, y_2][y_3, y_4] \ldots [y_{d-1}, y_d] \rangle,$$

where d is even, F is the free pro-p group with basis $y_1, \ldots, y_d$ and $y_1, y_2 \in \pi^{-1}(N \cap G_0)$ for the canonical epimorphism $\pi : F \to G_0$.

Proof. By Proposition 1 and by substituting for G a subgroup of finite index if necessary, we can assume that G has a minimal generating set $z_1, \ldots, z_d$ such that

$$w \equiv [z_1, z_2][z_3, z_4] \ldots [z_{d-1}, z_d] \text{ modulo } [[F, F], F],$$

where F is the free pro-p group on the set $z_1, \ldots, z_d$, G has the pro-p presentation

$$\langle z_1, \ldots, z_d \mid w \rangle$$

and

$$z_1, z_2 \in \pi^{-1}(N)$$

for the canonical map $\pi : F \to G$. Under the above assumptions (that are satisfied only after replacing G by a subgroup of finite index) we will show that G has a basis $y_1, \ldots, y_d$ that satisfies the conclusion of the theorem (for $G = G_0$).

Define

$$F_{k+1} = \overline{[F_k, F]} \text{ for } k \geq 1 \text{ and } F_1 = F, \text{ i.e. } F_{k+1} = \gamma_{k+1}(F)$$

and suppose we have found

$$z_1^{(i)}, \ldots, z_d^{(i)} \in F$$

such that

$$z_1^{(1)} = z_1, \ldots, z_d^{(1)} = z_d,$$

$$z_j^{(i+1)} \equiv z_j^{(i)} \text{ modulo } F_{i+1} \text{ for } 1 \leq j \leq d$$

and

$$w = [z_1^{(i)}, z_2^{(i)}][z_3^{(i)}, z_4^{(i)}] \ldots [z_{d-1}^{(i)}, z_d^{(i)}] f^{(i)} \text{ for some } f^{(i)} \in F_{i+2}. \tag{3}$$

Then $z_j^{(i+1)} = r_j^{(i)} z_j^{(i)}$ for some $r_j^{(i)} \in F_{i+1}$ and

$$[z_j^{(i+1)}, z_{j+1}^{(i+1)}] = [r_j^{(i)} z_j^{(i)}, r_{j+1}^{(i)} z_{j+1}^{(i)}] \equiv [r_j^{(i)}, z_{j+1}^{(i)}][z_j^{(i)}, r_{j+1}^{(i)}][z_j^{(i)}, z_{j+1}^{(i)}] \equiv$$

$$[r_j^{(i)}, z_{j+1}][z_j, r_{j+1}^{(i)}][z_j^{(i)}, z_{j+1}^{(i)}] \text{ modulo } F_{i+3}.$$

Thus

$$[z_1^{(i+1)}, z_2^{(i+1)}][z_3^{(i+1)}, z_4^{(i+1)}] \dots [z_{d-1}^{(i+1)}, z_d^{(i+1)}] =$$

$$[r_1^{(i)} z_1^{(i)}, r_2^{(i)} z_2^{(i)}] \dots [r_{d-1}^{(i)} z_{d-1}^{(i)}, r_d^{(i)} z_d^{(i)}] \equiv$$

$$[z_1^{(i)}, z_2^{(i)}][z_3^{(i)}, z_4^{(i)}] \dots [z_{d-1}^{(i)}, z_d^{(i)}] \beta(r_1^{(i)}, \dots, r_d^{(i)}) \text{ modulo } F_{i+3},$$

where

$$\beta(y_1, \dots, y_d) = [y_1, z_2][z_1, y_2] \dots [y_{d-1}, z_d][z_{d-1}, y_d].$$

By (3)

$$w \equiv [z_1^{(i+1)}, z_2^{(i+1)}][z_3^{(i+1)}, z_4^{(i+1)}] \dots [z_{d-1}^{(i+1)}, z_d^{(i+1)}] \text{ modulo } F_{i+3}$$

is equivalent to $r_1^{(i)}, \dots, r_d^{(i)}$ being a solution of the equation

$$\beta(r_1^{(i)}, \dots, r_d^{(i)}) \equiv f^{(i)} \text{ modulo } F_{i+3}. \tag{4}$$

Such a solution exists since by [20, Prop. 12.3.11] β induces a surjective homomorphism from the cartesian product of d copies of F_{i+1} to F_{i+2}/F_{i+3} if $i \geq 1$. But such a solution is not unique and our proof from now on will depend on manipulating different solutions.

We want to show by induction on i that $r_1^{(i)}$ and $r_2^{(i)}$ can be chosen from $\pi^{-1}(N)$, hence $z_1^{(i)}, z_2^{(i)} \in \pi^{-1}(N)$ for all i. Then we can define y_j as the limit of $z_j^{(i)}$ when i goes to infinity and by (3) we get

$$w = [y_1, y_2][y_3, y_4] \dots [y_{d-1}, y_d]$$

and as N is a closed subgroup $y_1, y_2 \in \pi^{-1}(N)$.

Note that since (4) is an equality modulo F_{i+3}, we are interested in $r_j^{(i)}$ only modulo F_{i+2} i.e. we are interested only in the image of $r_j^{(i)}$ in F_{i+1}/F_{i+2} and furthermore F_{i+1}/F_{i+2} is generated as an abelian pro-p group (i.e. as a $\mathbb{Z}_p$-module) by the images of the left normed commutators $[z_{j_1}, z_{j_2}, \dots, z_{j_{i+1}}]$ for $j_1, j_2, \dots, j_{i+1} \in \{1, 2, \dots, d\}$. Note that some of $j_1, \dots, j_{i+1}$ might be equal. If $\{j_1, j_2, \dots, j_{i+1}\} \cap \{1, 2\} \neq \emptyset$ using the fact that $z_1, z_2 \in \pi^{-1}(N)$ we get that $[z_{j_1}, z_{j_2}, \dots, z_{j_{i+1}}] \in \pi^{-1}(N)$. If $\{j_1, j_2, \dots, j_{i+1}\} \subseteq \{3, 4, \dots, d\}$ then by the Jacobi identity

$$[z_{j_1}, z_{j_2}, \dots, z_{j_{i+1}}, z_2] \in \prod_{t=1}^{i+1} [F_{i+1}, z_{j_t}] \subseteq \prod_{j=3}^{d} [F_{i+1}, z_j] \text{ modulo } F_{i+3}.$$

Thus the factors $[z_{j_1}, z_{j_2}, \dots, z_{j_{i+1}}]$ of $r_1^{(i)}$ with $\{j_1, j_2, \dots, j_{i+1}\} \subseteq \{3, 4, \dots, d\}$ can be moved from $r_1^{(i)}$ and distributed between $r_j^{(i)}$ for $j \geq 3$ i.e. we can suppose that $r_1^{(i)} \in \pi^{-1}(N)$. The same argument works for $r_2^{(i)}$. □

4 On subdirect products and virtual nilpotent quotients

We start with a pro-p version of a P. Hall theorem. Overline always denotes closure.

Lemma 5. *Let G be a free pro-p group and N be a non-trivial closed subgroup of G. Then there is a closed subgroup G_0 of finite index in G such that G_0 has a basis that contains at least one element of N.*

Proof. As N is non-trivial there is a subgroup G_0 of finite index in G such that the image of $N \cap G_0$ in $G_0/\overline{[G_0, G_0]G_0^p}$ is non-trivial. Note that any basis of $G_0/\overline{[G_0, G_0]G_0^p}$ as a vector space over $\mathbb{F}_p$ lifts to a basis of G_0 as a free pro-p group. □

We remind the reader that a pro-p HNN extension is proper if the base group embeds in the HNN-extension, see [16, p. 392].

Lemma 6. *Let G be a proper pro-p HNN extension with a base subgroup B and associated subgroup C such that B is topologically finitely generated and G is finitely presented as a pro-p group. Then C is topologically finitely generated.*

Proof. By [16, Prop. 9.4.2] there is a Mayer-Vietoris sequence

$$\ldots \to H_2(G, \mathbb{F}_p) \to H_1(C, \mathbb{F}_p) \to H_1(B, \mathbb{F}_p) \to \ldots$$

Since $H_2(G, \mathbb{F}_p)$ and $H_1(B, \mathbb{F}_p)$ are finite we get that $H_1(C, \mathbb{F}_p) \simeq C/\overline{[C, C]C^p}$ is finite i.e. C is topologically finitely generated. □

The following is a pro-p version of [3, Thm. 4.6]. Our proof relies significantly on the auxiliary results proved in the last section and the original proof of [3, Thm. 4.6].

Theorem 7. *Let G be a free pro-p group or a Demushkin group of depth $q = \infty$ and A be an arbitrary pro-p group. Let H be a closed subgroup of $A \times G$ that intersects G non-trivially and is finitely presented as a pro-p group. Then $H \cap A$ is (topologically) finitely generated.*

Proof. Let $\rho : A \times G \to G$ be the canonical projection. If $H \cap G$ has finite index in $\rho(H)$ then H contains $(H \cap A) \times (H \cap G)$ as a subgroup of finite index. Hence $(H \cap A) \times (H \cap G)$ and $L = H \cap A$ are finitely presented as pro-p groups and so are topologically finitely generated.

Assume now that $H \cap G$ has infinite index in $\rho(H)$. Note that $p(H)$ is either a Demushkin group of depth $q = \infty$ or a free pro-p group, thus without loss of generality we can assume that $\rho(H) = G$, hence G is (topologically) finitely generated. In particular $N = H \cap G$ is a non-trivial closed normal subgroup of infinite index in G and so N is a free pro-p subgroup. By Theorem 2 and Lemma 5 going down to a subgroup of finite index in G if necessary we can assume that either

1. $\rho(H) = G$ is a Demushkin group with a presentation $\langle y_1, y_2, \ldots, y_d \mid [y_1, y_2][y_3, y_4] \ldots [y_{d-1}, y_d] \rangle$ where d is even and $y_1, y_2 \in N$

or

2. $\rho(H) = G$ is a free pro-p group with a basis $y_1, \ldots, y_d$ and $y_1 \in N$,

where in both cases we have identified $y_1, \ldots, y_d$ with their images in G.

Note that there is a short exact sequence of groups

$$1 \to L \to H \to G \to 1.$$

Since G is finitely presented as a pro-p group and H is (topologically) finitely generated there is a finite subset $c_1, \ldots, c_n$ of L such that

$$c_1^H, \ldots, c_n^H \text{ topologically generate } L, \text{ hence}$$

$$c_1, \ldots, c_n \text{ (topologically) generate } L/\overline{[L, L]L^p} \text{ as a pro-}p\ \ \mathbb{F}_p[[G]] - \text{module}.$$

Assume first that we are in case 1, i.e. G is a Demushkin group. Pick $\widehat{y}_i \in \rho^{-1}(y_i) \cap H$; we can indeed take $\widehat{y}_1 = y_1$ and $\widehat{y}_2 = y_2$, thus

$$[\widehat{y}_1, \widehat{y}_2][\widehat{y}_3, \widehat{y}_4] \ldots [\widehat{y}_{d-1}, \widehat{y}_d] = c_0 \in L.$$

Let V be the closed subgroup of H topologically generated by L and the free pro-p group F_1 topologically generated by $y_2, \widehat{y}_3, \ldots, \widehat{y}_d$ (remember that a subgroup of infinite index in a Demushkin group has cohomological dimension less than 2, hence by [16, Thm. 7.7.4] the subgroup of G generated by $y_2, \ldots, y_d$ is a free pro-p group). As y_1 centralizes L and the quotient of G by the normal closed subgroup generated by y_1 is isomorphic to F_1 we have that

$$c_1, \ldots, c_n \text{ (topologically) generate } L/\overline{[L, L]L^p} \text{ as a pro-}p\ \ \mathbb{F}_p[[F_1]] - \text{module},$$

hence V is a split extension of L by F_1 and V is topologically generated by $c_1, \ldots, c_n$ and $y_2, \widehat{y}_3, \ldots, \widehat{y}_d$ (i.e. the images of these elements in $\widetilde{V} = V/\overline{[V, V]V^p}$ generate $\widetilde{V}$ as $\mathbb{F}_p$-vector space). Thus H has a pro-p presentation

$$\langle c_1, \ldots, c_n, y_1, y_2, \widehat{y}_3, \ldots, \widehat{y}_d \mid \text{ relations of } V, y_1^{-1} l y_1 = l \text{ for all } l \in L,$$

$$y_1^{-1} y_2 y_1 = y_2 [\widehat{y}_3, \widehat{y}_4] \ldots [\widehat{y}_{d-1}, \widehat{y}_d] c_0^{-1} \rangle.$$

In particular H is a proper pro-p HNN extension with a base V, a stable letter y_1 and associated subgroup $L \times \mathbb{Z}_p$, where $\mathbb{Z}_p$ is topologically generated by y_2. By Lemma 6, $L \times \mathbb{Z}_p$ is (topologically) finitely generated , hence L is (topologically) finitely generated as required.

Now suppose that we are in case 2, i.e. G is a free pro-p group. As in the proof of [13, Thm. 1] we pick $g_i \in \rho^{-1}(y_i) \cap H$ for $2 \leq i \leq m$. Let D be the closed subgroup of H topologically generated by L and the free pro-p F_2 group (topologically) generated by $g_2, \ldots, g_m$. As $t = y_1$ centralizes L we have

$$c_1, \ldots, c_n \text{ (topologically) generate } L/\overline{[L, L]L^p} \text{ as a pro-}p\ \ \mathbb{F}_p[[F_2]] - \text{module}.$$

Thus D is topologically generated by $c_1, \ldots, c_n$ and $g_2, \ldots, g_m$ (i.e. the images of these elements in $\widetilde{D} = D/\overline{[D, D]D^p}$ generate $\widetilde{D}$ as a $\mathbb{F}_p$-vector space). Note that L is a proper pro-p HNN-extension with a pro-p presentation

$$\langle D, t \mid t^{-1}bt = b \text{ for all } b \in L \rangle.$$

Then by Lemma 6, L is (topologically) finitely generated. □

The following result is a pro-p version of [3, Thm. 4.7].

Theorem 8. *Let $G_1, \ldots, G_n$ be free pro-p groups or Demushkin groups of depth $q = \infty$ and $H \subseteq D = G_1 \times G_2 \times \ldots \times G_n$ be a closed subdirect product (i.e. the projection of H to every factor G_i is surjective) that intersects every factor non-trivially. Suppose further that H is finitely presented as a pro-p group.*

Then there exist closed subgroups K_i of finite index in G_i such that

$$\gamma_{n-1}(K_i) \subseteq H \cap G_i \subseteq K_i$$

and the projection of H to any $j < n$ factors of D is again a finitely presented pro-p group.

Proof. The proof of Theorem 8 follows from Theorem 7 in exactly the same way as the proof of [3, Thm. 4.7] follows from [3, Thm. 4.4 & Them. 4.6]. □

5 On subdirect products of type FP_m

Lemma 9. *Let $Q_1, \ldots, Q_n$ be (topologically) finitely generated nilpotent pro-p groups and for all $1 \leq i \leq n$ let V_i be a (topologically) finitely generated pro-p $\mathbb{F}_p[[Q_i]]$-module that contains a free pro-p $\mathbb{F}_p[[Q_i]]$-submodule W_i. Suppose that $\widetilde{Q}$ is a closed subgroup of $Q = Q_1 \times \ldots \times Q_n$ such that $V_1 \widehat{\otimes}_{\mathbb{F}_p} \ldots \widehat{\otimes}_{\mathbb{F}_p} V_n$ is (topologically) finitely generated as a $\mathbb{F}_p[[\widetilde{Q}]]$-module. Then $\widetilde{Q}$ has finite index in Q.*

Proof. Note that $\mathbb{F}_p[[Q]] \simeq \mathbb{F}_p[[Q_1]] \widehat{\otimes}_{\mathbb{F}_p} \ldots \widehat{\otimes}_{\mathbb{F}_p} \mathbb{F}_p[[Q_n]] \simeq W_1 \widehat{\otimes}_{\mathbb{F}_p} \ldots \widehat{\otimes}_{\mathbb{F}_p} W_n =:$ W is a $\mathbb{F}_p[[\widetilde{Q}]]$-submodule of $V_1 \widehat{\otimes}_{\mathbb{F}_p} \ldots \widehat{\otimes}_{\mathbb{F}_p} V_n$ and $\mathbb{F}_p[[\widetilde{Q}]]$ is left and right Noetherian as an abstract ring. Since being (topologically) finitely generated over $\mathbb{F}_p[[\widetilde{Q}]]$ and being abstractly finitely generated over $\mathbb{F}_p[[\widetilde{Q}]]$ for a profinite $\mathbb{F}_p[[\widetilde{Q}]]$-module are the same we get that $\mathbb{F}_p[[Q]] = W$ is finitely generated (topologically or abstractly is the same) over $\mathbb{F}_p[[\widetilde{Q}]]$. Thus the Krull dimension of $\mathbb{F}_p[[Q]]$ is at most the Krull dimension of $\mathbb{F}_p[[\widetilde{Q}]]$.

On the other hand the Krull dimension of $\mathbb{F}_p[[H]]$ for a nilpotent pro-p group is the pro-p Hirsch length of H [1, Thm. A & Cor. C]. Then $\widetilde{Q}$ and Q have the same pro-p Hirsch length and so $\widetilde{Q}$ has finite index in Q. □

The following result has a version for abstract limit groups [10, Cor. 8].

Proposition 10. *Let G be a non-abelian pro-p group which is either a (topologically) finitely generated free pro-p group or a Demushkin group of depth $q = \infty$. Then for any prime number p and any natural number $k \geq 2$ the quotient $V = \gamma_k(G)/\overline{[\gamma_k(G), \gamma_k(G)]\gamma_k(G)^p}$ has a non-zero pro-p $\mathbb{F}_p[[Q]]$-submodule that is free, where $Q = G/\gamma_k(G)$.*

Proof. Consider first the case when G is a Demushkin group. Then there is an exact sequence of $\mathbb{Z}_p[[G]]$-modules coming from the presentation $\langle y_1, y_2, \ldots, y_d \mid [y_1, y_2][y_3, y_4] \ldots [y_{d-1}, y_d] \rangle$

$$\mathcal{P} : 0 \to \mathbb{Z}_p[[G]] \to \mathbb{Z}_p[[G]]^d \to \mathbb{Z}_p[[G]] \to \mathbb{Z}_p \to 0,$$

where $\mathbb{Z}_p$ is in dimension -1. Then

$$\mathcal{R} = \mathcal{P} \otimes_{\mathbb{Z}_p[[\gamma_k(G)]]} \mathbb{F}_p : 0 \to \mathbb{F}_p[[Q]] \to \mathbb{F}_p[[Q]]^d \to \mathbb{F}_p[[Q]] \to \mathbb{F}_p \to 0$$

has homology groups

$$H_2(\mathcal{R}) = H_2(\gamma_k(G), \mathbb{F}_p) = 0 \text{ and } H_1(\mathcal{R}) = H_1(\gamma_k(G), \mathbb{F}_p) \simeq V,$$

where the first equality comes from the fact that a subgroup of infinite index in a Demushkin group is a free pro-p group.

Note that Q is a torsion-free nilpotent pro-p group, $\mathbb{F}_p[[Q]]$ is a left and a right Noetherian ring without zero divisors [8, Cor. 7.25]. Then $\mathbb{F}_p[[Q]]$ is an Ore ring and has a classical ring of quotients, denoted by K. Note that K is an abstract ring (not a topological one) and it is flat as an abstract $\mathbb{F}_p[[Q]]$-module, hence $\otimes_{\mathbb{F}_p[[Q]]} K$ is an exact functor (here $\otimes$ is the abstract tensor product) and $V \otimes_{\mathbb{F}_p[[Q]]} K \simeq H_1(\mathcal{R}) \otimes_{\mathbb{F}_p[[Q]]} K \simeq H_1(\mathcal{R} \otimes_{\mathbb{F}_p[[Q]]} K) \simeq K^a$ for some non-negative integer a. Then

$$2 - d = \sum_i (-1)^i dim_K H_i(\mathcal{R} \otimes_{\mathbb{F}_p[[Q]]} K) =$$

$$\sum_i (-1)^i dim_K (H_i(\mathcal{R}) \otimes_{\mathbb{F}_p[[Q]]} K) = -dim_K (H_1(\mathcal{R}) \otimes_{\mathbb{F}_p[[Q]]} K)$$

and so

$$V \otimes_{\mathbb{F}_p[[Q]]} K \simeq K^{d-2}.$$

Since $d > 2$ we see that V has a subquotient (and hence a submodule) isomorphic to $\mathbb{F}_p[[Q]]$.

Now suppose that G is a free pro-p group. Then there is an exact complex of $\mathbb{Z}_p[[G]]$-modules

$$\mathcal{P} : 0 \to \mathbb{Z}_p[[G]]^d \to \mathbb{Z}_p[[G]] \to \mathbb{Z}_p \to 0$$

where d is the minimal number of generators of G. As in the first case the complex $\mathcal{R} = \mathcal{P} \otimes_{\mathbb{Z}_p[[\gamma_k(G)]]} \mathbb{F}_p$ has a unique non-trivial homology concentrated in dimension 1 and it is isomorphic to V. Thus $V \otimes_{\mathbb{F}_p[[Q]]} K \simeq H_1(\mathcal{R}) \otimes_{\mathbb{F}_p[[Q]]} K \simeq H_1(\mathcal{R} \otimes_{\mathbb{F}_p[[Q]]} K) \simeq K^{d-1}$ and since $d > 1$ we see that V has a subquotient (and hence a submodule) isomorphic to $\mathbb{F}_p[[Q]]$. ∎

Theorem 11. *Let each of $G_1, \dots, G_n$ be a free non-procyclic pro-p group or a non-abelian Demushkin group of depth $q = \infty$ and let $H \subseteq D = G_1 \times G_2 \times \dots \times G_n$ be a closed subdirect product such that H is finitely presented as a pro-p group and $H \cap G_i \neq 1$ for every $1 \le i \le n$. Then H is of type FP_m if and only if for every projection $p_{j_1,\dots,j_m} : D \to G_{j_1} \times \dots \times G_{j_m}$, the image $p_{j_1,\dots,j_m}(H)$ has finite index in $G_{j_1} \times \dots \times G_{j_m}$.*

Proof. By Theorem 8 and by replacing G_i by a subgroup of finite index for $1 \le i \le n$ if necessary we can assume that

$$\gamma_{n-1}(G_i) \subseteq H.$$

Let L be the direct product $\gamma_{n-1}(G_1) \times \dots \times \gamma_{n-1}(G_n)$ and $Q_i = G_i/\gamma_{n-1}(G_i)$. Thus L is a closed normal subgroup of H and $Q = H/L \subseteq D/L = Q_1 \times \dots \times Q_n$ is nilpotent and topologically finitely generated, hence of finite rank as a pro-p group. Then by [9, Thm. 3.2] H is of type FP_m if and only if the (continuous) homology groups $H_i(L, \mathbb{F}_p)$ are (topologically) finitely generated as $\mathbb{F}_p[[Q]]$-modules via the action of Q induced by conjugation for all $i \le m$.

Note that $\gamma_{n-1}(G_i)$ are pro-p subgroups of infinite index in G_i, hence are free pro-p groups. By the Kunneth formula and the fact that $H_k(\gamma_{n-1}(G_i), \mathbb{F}_p) = 0$ for $k \ge 2$ we get that for $i \le n$

$$H_i(L, \mathbb{F}_p) \simeq \oplus_{1 \le j_1 < j_2 < \dots < j_i \le n} H_1(\gamma_{n-1}(G_{j_1}), \mathbb{F}_p) \widehat{\otimes}_{\mathbb{F}_p} \dots \widehat{\otimes}_{\mathbb{F}_p} H_1(\gamma_{n-1}(G_{j_i}), \mathbb{F}_p)$$

where $\widehat{\otimes}$ is the completed tensor product and the action of Q on

$$H_1(\gamma_{n-1}(G_{j_1}), \mathbb{F}_p) \widehat{\otimes}_{\mathbb{F}_p} \dots \widehat{\otimes}_{\mathbb{F}_p} (H_1(\gamma_{n-1}(G_{j_i}), \mathbb{F}_p)$$

factors through the canonical map $h_{j_1,\dots,j_i} : Q_1 \times \dots \times Q_n \to Q_{j_1} \times \dots \times Q_{j_i}$. Thus if $h_{j_1,\dots,j_i}(Q)$ has finite index in $Q_{j_1} \times \dots \times Q_{j_i}$ for any $1 \le j_1 < j_2 < \dots < j_i \le n$ and $i \le m$ we get that $H_i(L, \mathbb{F}_p)$ is (topologically) finitely generated as a $\mathbb{F}_p[[Q]]$-module for $i \le m$. Hence H is of type FP_m as required.

For the converse suppose that H has type FP_m. Then by Lemma 9 and Proposition 10 $h_{j_1,\dots,j_i}(Q_1 \times \dots \times Q_n)$ has finite index in $Q_{j_1} \times \dots \times Q_{j_i}$, hence $p_{j_1,\dots,j_m}(H)$ has finite index in $G_{j_1} \times \dots \times G_{j_m}$. □

Corollary 12. *Let each of $G_1, \dots, G_n$ be a free non-procyclic pro-p group or a non-abelian Demushkin group of depth $q = \infty$ and let $H \subseteq D = G_1 \times G_2 \times \dots \times G_n$ be a closed subdirect product such that H has homological type FP_n and $H \cap G_i \neq 1$ for every $1 \le i \le n$. Then $(H \cap G_1) \times (H \cap G_2) \times \dots \times (H \cap G_n)$ is a subgroup of finite index in H and H has finite index in D.*

Proof. By Theorem 11, H has finite index in D, hence $H \cap G_i$ is a subgroup of finite index in G_i. □

Corollary 13. *Let each of $G_1, \dots, G_m$ be a free pro-p group or a Demushkin group of depth $q = \infty$, n a positive integer such that $n < m$ and G_i non-abelian exactly for $i \le n$. Let $H \subseteq D = G_1 \times G_2 \times \dots \times G_m$ be a closed subdirect product* *such that H has homological type FP_n and $H \cap G_i \neq 1$ for every $1 \leq i \leq m$. Then $(H \cap G_1) \times (H \cap G_2) \times \ldots \times (H \cap G_n) \times (H \cap (G_{n+1} \times \ldots \times G_m))$ is a subgroup of finite index in H.*

Proof. By the previous corollary $H_1 = (H \cap G_1) \times (H \cap G_2) \times \ldots \times (H \cap G_n)$ has a finite index in $H_0 = H \cap (G_1 \times G_2 \times \ldots \times G_n)$ and H_0 has finite index in $G_1 \times G_2 \times \ldots \times G_n$.

Let

$$p : D_1 = H_1 \times G_{n+1} \times \ldots \times G_m \to G_{n+1} \times \ldots \times G_m$$

be the canonical projection. Then $ker(p) = H_1 \subset H \cap D_1$ and so $H \cap D_1 = Ker(p) \times p(H \cap D_1) = H_1 \times (H \cap (G_{n+1} \times \ldots \times G_m))$ is a subgroup of finite index in H. □

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