GROUPS AND COMBINGS

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Abstract : We develop the ideas of combable and bicombable groups introduced by Cannon, Epstein, Holt, Patterson and Thurston [CEHPT] and others, generalizing the classes of automatic and biautomatic groups. These classes include Gromov's hyperbolic groups, and the fundamental groups of closed compact manifolds of non-positive curvature. We study quasiconvex subgroups of these groups and show that the results of Gersten and Short [GS1] for biautomatic groups can be extended to the class of bicombable groups. For instance it is shown that a nilpotent subgroup of a bicombable group is abelian by finite.

As an appendix we present an elementary introduction to quasiconvexity, and use the ideas to give a new proof of Howson's theorem: that the intersection of two finitely generated subgroups of a free group is finitely generated.

Resumé : Nous étudions les classes des groupes peignables et bipeignables, introduites par Cannon, Epstein, Holt, Patterson et Thurston [CEHPT], qui contiennent les classes des groupes automatiques et biautomatiques. Ces classes contiennent les groupes hyperboliques de Gromov, et les groupes fondamentaux des variétés fermées compactes de courbure non–positive. Nous étudions les sous–groupes quasiconvexes de ces groupes et nous montrons que les résultats de Gersten et Short [GS1] sur les groupes biautomatiques s'étendent aux groupes bipeignables. Par exemple, nous montrons qu'un sous–groupe nilpotent d'un groupe bipeignable est une extension finie d'un groupe abelien.

En appendice, nous présentons une introduction élémentaire aux idées de quasiconvexité dans un groupe libre, et nous utilisons ces idées pour donner une nouvelle démonstration du théorème de Howson : l'intersection de deux sous–groupes de type fini d'un groupe libre est de type fini.

Mots clefs: Geometric group theory, word hyperbolic group, Cayley graph, combable group

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Introduction

Current work in combinatorial group theory has led to the study of the geometry of a group, via the study of the Cayley graph as a metric space. Gromov's work [Gr] on hyperbolic groups and the work of Cannon [Ca] and Cannon et al. [CEHPT] on automatic groups are important examples of this type of approach. From this geometric point of view, words in the generators are identified with paths in the Cayley graph. Thus a word uniquely defines a path, once an initial vertex is specified.

In this article we shall study combings of groups, first introduced in [CEHPT] (see also [WPT] and [A]). Roughly speaking a combing is a set of representative words (written as products of the generators) for the elements of the group which have the following nice property: once an initial vertex is chosen, representative words which end at nearby elements of the group are uniformly close, as paths in in the Cayley graph. A combing becomes a bicombing if in addition we have the same property for representative words which begin at nearby vertices and end at the same vertex. Hyperbolic groups and the fundamental groups of closed, compact, non-positively curved manifolds are bicombable, as are groups acting freely and cocompactly on Euclidean buildings.

A (bi-)automatic group is a (bi-)combable group where in addition the set of representative words is a regular language in the free semigroup on the generators and their inverses. These groups are open to study via the theory of finite state automata, as initiated in [CEHPT]. We do not know of an example of a (bi-)automatic group which is not (bi-)combable. The results presented here are closely related to the results of [GS1] concerning biautomatic groups. In that article, certain subgroups of biautomatic groups are studied, and it is shown for instance that a nilpotent (or polycyclic) subgroup of a biautomatic group is a finite extension of a finitely generated abelian group. Here we shall establish the same result for bicombable groups, thus removing the necessity for considering regular languages and finite state automata.

The only important property of an automatic group which we do not know how to prove for combable groups is that an automatic group is not an infinite torsion group. Neither do we know whether or not bicombability is a *geometric* property, i.e. invariant under quasiisometry (see e.g. [Gh]) – not even whether a finite extension of a bicombable group is bicombable. We show however that the conjugacy problem is solvable for bicombable groups. Notice that Collins and Miller [CM] give an example of a group G with a solvable conjugacy problem which contains an index two subgroup H unsolvable conjugacy problem. They also give an example where the index two subgroup H has solvable conjugacy problem, but the overgroup G does not. Thus this problem is not geometric.

Alonso and Bridson [AB] have independently obtained most of the results presented here, using some related definitions of bicombability. Their aim is to find a geometric definition of semi-hyperbolicity to be to non-positively curved spaces what Gromov's definition of hyperbolicity is to negatively curved ones. In this way they hope to generalize many of Gromov's constructions and results, and to study, amongst other things, the structure of the group at infinity.

This paper is organised as follows. In the first section we develop the basic definitions of (bi-)combable groups, and show that the property of being combable is geometric (i.e. is a quasiisometry invariant). The definition we use differs from that given in [CEHPT] in that we do not require representative words to be quasi-geodesic. In the second section we show that combable groups are finitely presented and have solvable word problem. Bicombable groups are shown to have solvable conjugacy problem. Here we use some recent ideas of A. Casson, J. Stallings and S.M. Gersten concerning isodiametric inequalities. The following section contains a discussion of quasiconvex subgroups of (bi-)combable groups, following [GS1]. The useful results here are that a quasiconvex subgroup of a (bi-)combable group is (bi-)combable (§3.2), and that the intersection of two quasiconvex subgroups is also quasiconvex (§3.4). This last result takes the place of analogous result of [GS1] for regular subgroups. In the final section we show how to generalize the results of [GS1] to the class of bicombable groups, using the translation length as in [GS1]. We finish with a new proof, due to S.M. Gersten, of Yau's theorem, that an abelian subgroup of a fundamental group of a non-positively curved closed Riemannian manifold is finitely generated (§4.5).

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Section 1 Definitions and elementary properties

Let G be a finitely generated group, and let X be a finite set of generators for G. By adjoining a set of formal inverses, we obtain a set of monoid generators $\mathcal{A} = X \cup X^{-1}$. As we wish to allow non-reduced words here, we shall consider the free monoid on $\mathcal{A}$, denoted $\mathcal{A}^\star$, of all finite words on the generators in X and their inverses. The free group is naturally contained in $\mathcal{A}^\star$, and the map $\mu : \mathcal{A}^\star \to G$ is well defined. As usual, we denote the length of a word $w \in \mathcal{A}^\star$ by $\ell(w)$, and we write

$$|g|_X = \min\{\ell(w) \mid w \in \mathcal{A}^\star,\ \mu(w) = g\}.$$

The Cayley graph $\Gamma_X(G)$ of G with respect to a set of generators X has a vertex for each element of G, and an oriented edge from g to $g\mu(x)$ for each $g \in G$ and for each $x \in X$. We make the Cayley graph into a metric space by assigning length 1 to each edge, and defining the distance $d(g,h)$ between the vertices corresponding to $g, h \in G$ to be the infimum of the lengths of (non-directed) paths joining them. Thus the distance in the Cayley graph is $d(g,h) = |g^{-1}h|_X$. It is clear that with this definition of distance, the group G acts on the Cayley graph $\Gamma_X(G)$ on the left by isometries.

We shall frequently identify the word $w \in \mathcal{A}^\star$ with the path

$$w : ([0,\infty), 0) \to (\Gamma_X(G), 1)$$

where w is a local isometry of $[0, \ell(w)]$ onto the path in $\Gamma_X(G)$ based at the identity which spells out the letters of w, and $w(t) = \mu(w)$ for $t > \ell(w)$; i.e. if $w = a_1 \dots a_n$, then $w([i-1, i])$ is the edge labelled a_i based at the vertex labelled $\mu(a_1 \dots a_{i-1})$. (A negative exponent indicates that the edge with this letter is traversed in the opposite sense to its given orientation.)

Definition

A map $\sigma : G \to \mathcal{A}^\star$ is called a *section* if it is a right inverse to the natural surjection μ, i.e. for all $g \in G$, $\mu \cdot \sigma(g) = g$.

A section $\sigma : G \to \mathcal{A}^\star$ is called a *combing* if there is a positive constant K_1 such that for each $g \in G$ and $x \in X$ the following condition is satisfied:

C1) $d(\sigma(g)(t), \sigma(g\mu(x))(t)) \le K_1$.

In this case we say that the words $\sigma(g)$ and $\sigma(g\mu(x))$ are uniformly K_1-close (or K_1 fellow travellers).

If each word $\sigma(g)$ is a geodesic, i.e. if $\ell(\sigma(g)) = |g|_X$, then the combing is said to be *geodesic* . More generally, we say that the combing is *short* if, in addition to the condition C1, there is a constant K_2 such that for all $g \in G$, and all $x \in X$, the combing also satisfies:

C2) $|\ell(\sigma(g\mu(x))) - \ell(\sigma(g))| \le K_2$.

In order to consider a symmetric property concerning words which begin close and end at the same vertex, we need to consider the left action of G on $\Gamma_X(G)$.

For $h \in G$ and $w \in \mathcal{A}^\star$, we use $h \cdot w$ to denote the path w translated to begin at the vertex corresponding to the element h - i.e. the image of w under the left action of G on $\Gamma_X(G)$. A combing is called a *bicombing* if in addition to C1) we have for each $g \in G$ and $x \in X$

C3) $d(\mu(x) \cdot \sigma(g)(t), \sigma(\mu(x)g)(t)) \le K_1$.

As before, we say that the bicombing is short if C1, C2, and C3 old, and in addition

C4) $|\ell(\sigma(\mu(x)g)) - \ell(\sigma(g))| \leq K_2$.

If $K \geq \max K_i$, we shall often call a (bi-)combing which satisfies C1 and C2 (and C3, C4) a K-(bi-)combing.

A group is said to be *(bi-)combable* if it has a (bi-)combing satisfying C1 (and C3). Alonso [A] calls any section $G \to \mathcal{A}^\star$ a combing, and calls a section with the above condition C1 a quasi-Lipschitz combing.

Examples 1.1

0) Clearly a finite group has a bicombing (for instance take the set of all elements as a set of generators).

1) Consider the infinite cyclic group generated by x; clearly the set of representatives $\{x^m \mid m \in \mathbf{Z}\}$ gives a geodesic bicombing with constant 1. If however we take as representatives $\{x^m(x^2x^{-2})^5\}$ we get a bicombing which is no longer geodesic, but is short, and we may take constants $K_1 = 2$, and $K_2 = 1$. If we now consider

$$\{x^m(x^2x^{-2})^{3^m}\}$$

we obtain a bicombing which is not short, with $K_1 = 3$. Finally notice that

$$\{(x^2x^{-2})^{3^m}x^m\}$$

is not a combing.

2) For a finitely generated free group, the set of freely reduced words forms a bicombing, with constant 1.

3) A symmetric combing ($w = \sigma(g)$ implies $w^{-1} = \sigma(g^{-1})$) is a bicombing.

4) Following Gromov [Gr], say that a group is δ-hyperbolic if geodesic triangles in the Cayley graph are δ-slim (each side is contained in a δ-neighbourhood of the other two). For a δ-hyperbolic group G choose a geodesic representative $\sigma(g)$ for each element $g \in G$. For each generator x, there is a geodesic triangle with sides labelled $\sigma(g), \sigma(gx)$ and x for each $g \in G$. The sides $\sigma(g)$ and $\sigma(gx)$ are uniformly $2(\delta+1)$-close. This means that we have a combing. As the set of all geodesics is symmetric, by considering the triangle with sides $x, \sigma(\mu(x)g), \mu(x) \cdot \sigma(g)$ it is easily seen that in fact we have a bicombing with constant $2(\delta+1)+1$.

5) The finitely generated free abelian group $\mathbf{Z}^n$ has a bicombing $\{(m_1, \ldots, m_n)\}$ with constant 1, where the path has the form $x_1^{m_1} \ldots x_n^{m_n}$.

6) For $\mathbf{Z}^2$, generated by x, y, the section

$$\{x^n y^m \mid |n| \geq |m|\} \cup \{y^m x^n \mid |n| < |m|\}$$

is not a combing — the words $x^n y^n, y^{n+1}x^n$ are not uniformly k-close for $k < 2|n|$.

7) For the Heisenberg group $\{a^k b^m c^n\}$ is a section, but is not a combing, as $a^k b^m c^n.a = a^{k+1} b^m c^{n-m}$ where

$$a = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} b = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} c = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

In fact this group has no short combing as it does not satisfy a quadratic isoperimetric inequality (see section 2 below, and [CEHPT], [G1]).

8) Let G be a group acting freely and cocompactly on a simplicial complex $\widetilde{K}$ of non-positive curvature. Then there exist unique geodesic arcs between points of $\widetilde{K}$, and a finite sided fundamental domain D for the action of G. Taking as generators for G the reflections in the sides of this fundamental domain, the Cayley graph of G is quasiisometric to $\widehat{K}$, with vertices in the interior of the fundamental domain. A bicombing for the group can be obtained by taking as a representative corresponding to the image gD of the fundamental domain, a shortest word $w = a_1 \ldots a_n$ such that $\{a_1 \ldots a_j D \mid j = 1 \ldots n\}$ covers the unique geodesic from the point $v \in D$ to $gv \in gD$. (This is just the construction given by Milnor in [M], previously studied by Svarc [Sv].)

Notes

1) By an easy induction argument, if σ is a combing, then $d(\sigma(g)(t), \sigma(h)(t)) < K_1 |g^{-1}h|_X$.

2) Given a combing, we can always alter it on a finite subset, at the expense of possibly increasing the value of the constant K_1. So we may assume, if we wish that the representative of the identity in a combing is the empty word.

Definition Following Cannon et al. [CEHPT], we say that a group G generated by the finite set X is *(bi-)automatic* if it is (bi-)combable with a section σ which is a regular language in $\mathcal{A}^\star$ (see also [WPT], [BGSS]). (The pair $(\mathcal{A}, \sigma(G))$ forms an automatic structure for G.) This definition is equivalent to the definition in terms of comparator automata for multiplication by generators, by the construction of the so-called standard automata, and by the fact that automaticity is independent of the set of (semigroup) generators chosen.

Thus hyperbolic groups, small cancellation groups and finitely generated abelian groups are bicombable, being biautomatic (see [GS1;2]). The alternating knot groups are also biautomatic, so that the torus knot groups $\langle a, b \mid a^p = b^q \rangle$, with p, q coprime, are bicombable.

Before establishing some of the closure properties of the class of (bi-) combable groups, we need to prove some lemmas. We first show that the existence of a (bi-)combing is independent of the set of generators chosen. This proof is virtually identical to that of the analogous statement for automatic groups (see [CEHPT]).

We now show that the combability properties are quasiisometry invariants (as in [Gr], [DG],[CEHPT],[Gh]). We recall the definition of a quasiisometry: Let V, W be metric spaces. A function $f : V \to W$ is a (C, ϵ)-*quasiisometry* if

for all $v_1, v_2 \in V$,

$$\frac{1}{C} d_V(v_1, v_2) - \epsilon \leq d_W(f(v_1), f(v_2)) \leq C d_V(v_1, v_2) + \epsilon.$$

Notice that such a map need not even be continuous; consider the quasiisometry from the reals to the integers given by taking integer part.

Two metric spaces V and W are said to be quasiisometric if there are (C, ϵ)-quasiisometries $f : V \to W$ and $f' : W \to V$ such that for all $v, v' \in V$, $w, w' \in W$,

$$\begin{aligned} d(f' \circ f(v), v) &\le C \\ d(f \circ f'(w), w) &\le C \\ d(f(v), f(v')) &\le Cd(v, v') + \epsilon \\ d(f'(w), f'(w')) &\le Cd(w, w') + \epsilon. \end{aligned}$$

When considering Cayley graphs of groups, the additive constant ϵ can be neglected as we need only establish the conditions of the quasiisometry for the vertices.

Thus $\mathbf{R}^n$ and $\mathbf{Z}^n$ are easily seen to be quasiisometric, and the Cayley graphs of a group with respect to different finite generating sets are quasiisometric. Also a group and a finite index subgroup are easily seen to be quasiisometric. The Cayley graph of a the fundamental group of a closed compact manifold is quasiisometric to the universal cover of the manifold.

Proposition 1.2.
If $\Gamma_X(G)$ and $\Gamma_Y(H)$ are quasiisometric, and G is combable, then H is too.

Proof.
We begin by supposing that Y contains a generator z such that $\mu(z) = 1$.

Let $f : \Gamma_X(G) \to \Gamma_Y(H)$ and $f' : \Gamma_Y(H) \to \Gamma_X(G)$ be the quasiimetries satisfying the above conditions. Letting $\mathcal{A} = X \cup X^{-1}$ and $\mathcal{B} = Y \cup Y^{-1}$, and let $\sigma : G \to \mathcal{A}^\star$ be a combing with constant K_1. We define a section $\sigma' : H \to \mathcal{B}^\star$ as follows.

Let $\sigma(g) = a_1 \dots a_n$; define $\sigma_1(g) = A_1 \dots A_n$ where A_i is a shortest word in $\mathcal{B}^\star$ such that $f(a_1 \dots a_{i-1})A_i = f(a_1 \dots a_i)$ in H. By the conditions on f, $\ell(A_i) \le C + \epsilon = C'$. Now pad the end of each word A_i with the letter z until the length is exactly C'; i.e. replace A_i by $A_i z^m$ where $m = C' - \ell(A_i)$. Call the resulting word B_i, and define $\sigma_2(g) = B_1 \dots B_n$.

If $d(f(g), f(g')) \le 2C + 1$, then

$$\begin{aligned} d(g, g') &\le d(g, f' \cdot f(g)) + d(f' \cdot f(g), f' \cdot f(g')) + d(f' \cdot f(g'), g') \\ &\le C + C(2C + 1) + \epsilon + C = 2C^2 + 3C + \epsilon. \end{aligned}$$

The combing condition C1 and induction give that $\sigma(g)$ and $\sigma(g')$ are uniformly $K(2C^2 + 3C + \epsilon)$-close in $\Gamma_X(G)$. It follows that $\sigma_2(g)$ and $\sigma_2(g')$ are paths in $\Gamma_Y(H)$ which are at most $C'K(2C^2 + 3C + \epsilon)$ apart at points $\sigma_2(g)(mC')$ for integer multiples mC', and thus in general are uniformly $C'K(2C^2 + 3C) + 2C'$-close.

For $h \in H$, define $\sigma'(h) = \sigma_2(f'(h)).v(h)$, where v is the label on some shortest path in $N_C(h)$ from $f(f'(h))$ to h; thus $\ell(v(h)) \le C$.

If $d(h, h') = 1$, then $d(f(f'(h)), f(f'(h'))) \le 2C + 1$, and so, by the above, $\sigma'(h)$ and $\sigma'(h')$ are uniformly $C'K(2C^2 + 3C + \epsilon) + 2C' + 2C$ close. (The extra $2C$ is to take into account the appended words $v(h), v(h')$, each of length $\le C$.).

Now if there is no word $z \in Y$ with $\mu(z) = 1$, adjoin such an element, proceed as above, then replace every second occurence of z by yy^{-1} for some $y \in Y$. This will make words at most 1 out of step with their previous parametrization, so add 2 to the constant of uniformity.

If the combing σ satisfies the condition C2, then in the same way we see that σ' satisfies the same condition, for some other constant. ∎

Any automatic structure $(\mathcal{A}, L)$ for an automatic group G gives rise to naturally to an automatic structure $(\mathcal{A}, L')$ where L' is a combing, and $L' \subset L$. In this case Cannon et al. [CEHPT] have shown that the regular language L' consists of quasigeodesics, i.e. there are constants C, ϵ such that each path $w \in L'$ is a (C, ϵ)-quasiisometry of $[0, \ell(w)]$ to $\Gamma_X(G)$. Clearly this is a short combing. In the case of combable groups we do not have such a strong restriction on the section.

Proposition 1.3.
1) The free product of two combable groups is combable.
2) The free product of two groups with short bicombings has a short bicombing.
3) The direct product of two groups with short (bi-)combings has a short (bi-)combing.
4) A retract of a (bi-)combable group is (bi-)combable.

Proof.
1) Let $\sigma_G : G \to \mathcal{A}^\star$ and $\sigma_H : H \to \mathcal{B}^\star$ be combings for the groups G, H . An element $g \in G\star H$ has a unique normal form $g = a_1b_1 \dots a_nb_n$. Then the section $\sigma_{G\star H} : G \star H \to (\mathcal{A} \cup \mathcal{B})^\star$ given by $\sigma_{G\star H}(a_1b_1 \dots a_nb_n) = \sigma_G(a_1)\sigma_H(b_1) \dots \sigma_G(a_n)\sigma_H(b_n)$ is easily seen to be a combing.

2) To obtain a bicombing, we need the condition C3, so that $|\ell(\sigma_G(\mu(x)a_1)) - \ell(\sigma_G(a_1))|$ is bounded, for $x \in X$.

3) The section $\sigma_{G\times H} : G\times H \to (\mathcal{A}\cup\mathcal{H})^\star$ given by $\sigma_{G\times H}(g, h) = \sigma_G(g)\sigma_H(h)$ is a bicombing, with constant $2\max\{K_G, K_H\}$. Notice that in order to get the uniform bound on the distance between the words $\sigma_{G\times H}((g, h)\mu(x))$ and $\sigma_{G\times H}((g, h))$ for $x \in X$, we need the fact that

$$|\ell(\sigma_G(g)) - \ell(\sigma_G(g\mu(x))|$$

is bounded. In other words, the condition C2) is required even to show that the direct product is combable.

4) Let $r : G \to H$ be a retraction of the (bi-)combable group G onto H. It is clear that a retract of a finitely generated group is finitely generated.

Take as generators for H the set X of generators for G, and define $\mu_H(x) = r(\mu_G(x))$. Then $\sigma|_H$ is a bicombing, as r does not increase distance.

Alternatively, select as generators for G the union $X \cup Y$, where X is a finite set of generators for H. Put $\mathcal{A} = X \cup X^{-1}$, and $\mathcal{B} = Y \cup Y^{-1}$. Let $r : G \to H$ be the retraction. For each $y_i \in Y$ let $\phi(y_i)$ denote a word in $F(X)$ such that $\mu(\phi(y_i)) = r(\mu(y_i))$ in H. Let $\phi' : (\mathcal{A} \cup \mathcal{B})^\star \to \mathcal{A}^\star$ denote the map induced by the identity on X, and ϕ on Y. If $\sigma : G \to (\mathcal{A} \cup \mathcal{B})^\star$ is a (bi-)combing, then $\phi' \cdot \sigma|_H : H \to \mathcal{A}^\star$ is a section, and as in lemma 1.1, this section can be made into one which is a bicombing. To do this, let N be the length of the longest element $\phi(y_i)$. Read along a word $\phi'(\sigma(h))$ omitting subwords $\phi(y_i)$ until more than $N - 1$ letters have been omitted (i.e. until $\sum_{i=1}^{J} \ell(\phi(y_i)) > N - 1$). Include the following subwords $\phi(y_i)$ until fewer than N letters have been omitted (i.e. until $\sum_{i=1}^{J} \ell(\phi(y_i)) - \sum_{j=J+1}^{J'} \phi(y_i)) < N$). Continue in this way. The words obtained form a bicombing with constant $K + \frac{N}{2}$. ■

Questions Is every combable group bicombable? Is a finite extension of a bicombable group bicombable? Is a free product of two bicombable groups amalgamating a finite subgroup bicombable?

Proposition 1.4.
The amalgamated product of combable groups amalgamating a finite subgroup is itself combable.

Proof. Let $G = A \star_C B$, with C finite. By the normal form theorem for amalgamated products, we can write each element of the product in the form $g = a_1 b_1 \ldots a_n b_n c$ where $a_i \in S$, $b_i \in T$ where S and T are transversals for C in A and B respectively. We choose as generators for G the entire set C, together with a finite set X_S (resp. X_T) of elements of S (resp. T) to give a set of generators for A (resp. B). Let σ_A and σ_B be combings with respect to these sets of generators. Set $\sigma(g) = \sigma_A(a_1)\sigma_B(b_1) \ldots \sigma_A(a_n)\sigma_B(b_n)c$. This is easily seen to be a combing, as for each $c \in C$, and each generator $s \in X_S$, there is a word $w(c,s) \in F(X_S \cup C)$ and an element $c' \in C$ such that $\mu(cs) = \mu(w(c,s)c')$. Similarly for multiplication on the right by en element of X_T. The constant for the combing depends thus on the maximum length of the words $w(c,s)$ (and the analoguous words $w(c,t)$). ■

Section 2 The word and conjugacy problems

We begin by following the proof for automatic groups of the solvability of the word problem to show that combable groups are finitely presented.

Proposition 2.1. ([CEHPT])
Combable groups are finitely presented.

Proof. Let $\sigma : G \to \mathcal{A}^\star$ be a combing for the group G.

In order to decide whether a word $w = b_1 \dots b_n \in F(X), b_i \in X \cup X^{-1}$ represents the trivial element of G, we show how to obtain the representative $\sigma(b_1 \dots b_{j+1})$ from $\sigma(b_1 \dots b_j)$. As these two words are uniformly K close, there is a sequence of words v_t such that

$$\mu(\sigma(b_1 \dots b_{j+1})(t)) = \mu(\sigma(b_1 \dots b_j)(t) v_t)$$

where in addition $\ell(v_t) < K$ for all t. This means that there is a sequence of words $r_1, \dots, r_s$ and $p_1, \dots, p_s \in F(X)$ with $\ell(r_i) \leq 2K+2$, $\mu(r_i) = 1$, $p_i = \sigma(b_1 \dots b_j)(i)$,

$$\sigma(b_1 \dots b_j) b_{j+1} \sigma(b_1 \dots b_{j+1})^{-1} = \prod_{i=1}^{N} p_i r_i p_i^{-1},$$

where $N = \max\{\ell(\sigma(b_1 \dots b_j)), \ell(\sigma(b_1 \dots b_{j+1}))\}$.

Figure 1

■

We follow some recent ideas due to A. Casson, J. Stallings, S.M. Gersten and maybe others too (see [S], [G2]), to see that combable groups have a solvable word problem.

Let $\mathcal{P} = \langle X \mid R \rangle$ be a finite presentation for the group G. Let

$$R_n = Sbgp_{F(X)}\{uru^{-1} \mid \ell(u) \leq n\} .$$

Definition

We say that the function $f : \mathbf{N} \to \mathbf{R}$ is an isodiametric function for $\mathcal{P}$ if for all $w \in F(X)$ such that $\mu(w) = 1$ in G, we have

$$w \in R_{f(\ell(w))}.$$

An alternative formulation due to S. M. Gersten [G2] is to consider van Kampen diagrams for $w = 1$ in G. The 1-skeleton of such a diagram maps into the Cayley graph $\Gamma_X(G)$ in such a way that the boundary is the path w. Then f is an isodiametric function if there is a diagram for $w = 1$ such that each vertex on the diagram is at distance at most $f(\ell(w))$ from the base point.

When f is a linear function, S.M. Gersten has introduced the notation $\mathcal{P}$ satisfies $ID^\star(\alpha)$ when there is a diagram for $w = 1$ such that each vertex is at distance at most $\alpha\ell(w)$ from the base vertex. Gersten has also obtained the results 2.3 and 2.4 below, and many other interesting results concerning the $ID^\star$ condition. In particular he has shown that all compact 3-manifolds satisfying Thurston's geometrization conjecture have fundamental groups with linear isodiametric functions.

Proposition 2.2.

Let $\mathcal{P}$ be a finite presentation for the group G.

Then $\mathcal{P}$ has a recursive isodiametric function if and only if G has a solvable word problem.

Proof.

Sufficiency: We show that the existence of a recursive isodiametric function means that we can construct enough of the Cayley graph $\Gamma_X(G)$ to see whether w does or does not represent the identity element of G.

Let f be a recursive isodiametric function. Construct the finite, labelled tree which is ball of radius $f(\ell(w)/2) + \max \ell(r)$ about the origin in the the Cayley graph of $F(X)$. Write each relation $r \in R$ as $r' = r''$; this can be done in a finite number of different ways. For each word $u \in F(X)$ of length at most $f(\ell(w))$, and for each relation $r' = r''$, identify the vertices defined by the paths from the identity vertex labelled ur' and ur''. When this is done, all loops in the Cayley graph corresponding to relations in $R_{f(\ell(w))}$ have been constructed.

Tracing out the word w on the finite labelled graph we have constructed, a closed loop is obtained if and only if w represents the trivial element of G.

Necessity: The path corresponding to a word of length at most n lies inside B_n, the ball about the identity vertex of radius n in $\Gamma_X(G)$. Now solve the word problem for all words of length $\leq n$, and let $f(n) = \max \ell(u)$ where the word u appears in some term uru^{-1} in some expression $w = \prod u_i r u_i^{-1}$ for some word w with $\mu(w) = 1$ and $\ell(w) \leq N$. As there are a finite number of such words, f is indeed a recursive function, as required. ■

Change of generating set will in general alter an isodiametric function, but not by much as we shall now see. First we define an equivalence relation between functions by $f \sim g$ if:

$$\text{there are constants } A, B, C \geq 0, \text{ such that } \quad f(t) = Ag(Bt + C) + Dt + E.$$

The following proposition is then not hard to establish:

Proposition 2.3.

Let $\mathcal{P} = \langle X \mid R \rangle$ be a finite presentation of the group G.

1) If G is combable, then $\mathcal{P}$ has a linear isodiametric function.

2) If H is quasiisometric to G then H has an equivalent isodiametric function.

3) If G has a geodesic combing with respect to X, then there is a constant $d \leq 0$ such that $f(t) = t/2 + d$ is an isodiametric function for $\mathcal{P}$.

Proof. Let $\sigma : G \to \mathcal{A}^\star$ be a combing, with constant K_1, and suppose that the representative of the trivial element is the empty word.

1) Initially let us suppose that the set of relators R contains the set R' of all words in $F(X)$ of length at most $2K_1+2$ which are trivial in G. We use the notation of the proof of 2.1. There we saw how to express a word w representing the identity element of G as a product of relators in R. Notice that each relator which occurs in the diagram for

$$\sigma(b_1 \dots b_j) b_{j+1} \sigma(b_1 \dots b_{j+1})^{-1}$$

is at distance at most K_1 from a vertex on the path $\sigma(b_1 \dots b_{j-1})$. Thus each vertex of the diagram constructed there is at distance at most $C' = K_1(\ell(w)/2 + 1) + 2$ from the boundary, and hence the presentation satisfies $ID^\star(2K_1)$.

If the set R does not contain R' then it suffices to find a diagram for each element of R', using the relators in R. Now add the maximum distance C'' of a vertex in one of these diagrams to the constant C' above.

2) Let g be an isodiametric function for $\Gamma_X(G)$. Let $f : \Gamma_X(G) \to \Gamma_Y(H)$ and $f' : \Gamma_Y(H) \to \Gamma_X(G)$ be (C, ϵ)-quasiisometries such that $f \circ f'$ and $f' \circ f$ satisfy the conditions given before 1.2. Let $w \in F(Y)$ be a word representing the trivial element of H, which we regard as a path in $\Gamma_Y(H)$. Then $f'(w)$ is a path in $\Gamma_X(G)$, and has length at most $C\ell(w) + \epsilon$. There is a diagram D for the word defined by $f'(w)$ and each vertex is at distance at most $g(\ell(f'(w))$ from the base point. We now map the entire diagram by f to $\Gamma_X(G)$. The loops corresponding to relators of G map to loops in $\Gamma_Y(H)$ whose length is bounded (by $C \max \ell(r) + \epsilon$) and hence, as in 1), each such loop bounds a diagram where each vertex is at bounded distance C'' from its base vertex. The word $f(f'(w))$ and the word w bound an annular diagram, which we can regard as consisting of $\ell(w)$ regions, each of boundary length $1+3C+\epsilon$. As before this gives an additive constant C''' for the distance of a vertex in of some diagram for one of these finitely many regions to some base vertex. Thus we obtain $Cg(Ct + \epsilon) + C\epsilon + \max\{C'', C'''\}$ as an isodiametric function for H.

3) When the combing is geodesic, it is clear in the proof of 2.1 that each relator in R' used in the diagram is at distance at most $\ell(w)/2$ from the base vertex, by travelling along $\sigma(b_1 \dots b_j)$.

It follows immediately from 2.2 and 2.3 that:

Corollary 2.4.
The word problem for a combable group is solvable.

A related idea concerns a combinatorial idea of area, developped by Gersten in [G1].

Definition

We say that $f : \mathbf{N} \to \mathbf{R}$ is a Dehn function for the finite presentation $\mathcal{P} = \langle X \mid R \rangle$ if for all $w \in F(X)$ such that $\mu(w) = 1$, we can find $p_i \in F(x)$, and $r_i \in R$, $\epsilon_i = \pm 1$, and $N \leq f(\ell(w))$ such that

$$w = \prod_{i=1}^{N} p_i r_i^{\epsilon_i} p_i^{-1} \text{ in } F(X) \quad .$$

Gersten has shown in [G1] that different finite presentations of a group have equivalent isoperimetric Dehn functions. Quasiisometric groups also have equivalent Dehn functions (a proof can be reconstructed from the proof of 2.3.2 above, or see [A2], or [Ba]). Gromov [Gr] has shown that hyperbolic groups are characterized as being those groups which have a finite presentation with a linear Dehn function (for an alternative proof see the appendix of [GS2]).

Proposition 2.5. (*cfr.* [CEHPT])
A group which has a short combing has a quadratic Dehn function.

Proof. Let σ be a short combing. Using the notation of the proof of Proposition 2.1,

$$\ell(\sigma(b_1 \dots b_{j+1})) \leq \ell(\sigma(b_1 \dots b_j)) + K_2.$$

Thus if $\mu(w) = 1$, the word w can be expressed as a product of at most $K_2\ell(w)(\ell(w)+1)/2$ conjugates of the relators r_i, where in addition the conjugating words are of length at most $\ell(w)K_2$. ■

We do not know whether the conjugacy problem is solvable for combable groups – the result is not know even if one retricts to automatic groups. In [GS2] it is shown that the conjugacy problem is solvable for biautomatic groups. Similar ideas as are used there and in [GS1] can be used as follows to show:

Proposition 2.6.
The conjugacy problem for bicombable groups is solvable.

Proof. Let $\sigma : G \to \mathcal{A}^\star$ be a bicombing. Let $x, y \in F(X)$ be words representing conjugate elements of G. Let g be a conjugating element such that $g\mu(x) = \mu(y)g$; we shall show that such a g can be found whose length is bounded by some function of $\max\{\ell(x), \ell(y)\}$. By the bicombing properties, $\sigma(g)$ is uniformly $K\ell(x)$ close to $\sigma(g\mu(x))$ which in turn is uniformly $K\ell(y)$-close to $\mu(y) \cdot \sigma(g)$ (as $\sigma(g\mu(x)) = \sigma(\mu(y)g)$).

Let $\sigma(g) = a_1 \dots a_n$. There is thus a sequence $x = \gamma_0, \gamma_1, \dots, \gamma_n = y$ of words in $F(X)$ such that $\gamma_i = a_i\gamma_i a_i^{-1}$ and each $\ell(\gamma_i) < K\ (\ell(x) + \ell(y)) = M$. Notice that if $\gamma_i = \gamma_j$ for some $i \neq j$, then $\mu(a_1 \dots a_i a_{j+1} \dots a_n)$ conjugates $\mu(x)$ to $\mu(y)$.

If M is the number of elements of G of length at most N, then there is a word w' with $\ell(w') < M$ such that $\mu(w'x) = \mu(yw')$.

It now follows from the solvability of the word problem that the conjugacy problem is solvable, as given x, y, there are a finite number of words $w'xw'^{-1}y^{-1}$ to check. ■

Notice that it is not necessary here for the bicombing to be short.

Section 3 Quasiconvex subgroups

In order to show that the centralizer of an element is a bicombable group in its own right, we first introduce the definition of a *quasiconvex* subgroup.

Definition

Let $s : G \to \mathcal{A}^\star$ be a section (not necessarily a combing). A subset A of G is said to be s-*quasiconvex* if there is a positive constant C such that for all $a, b \in A$, the path $a \circ s(a^{-1}b)$ lies in a C-neighbourhood of A in the Cayley graph $\Gamma_X(G)$. That is, all paths which are in s, between vertices of A, lie close to the subset A. When A is a subgroup, it suffices to consider paths based at the identity vertex, and A is an s-quasiconvex subgroup when for all $a \in A$, the path $s(a)$, based at the identity, lies in a C-neighbourhood of A.

Clearly any finite subset of a group is quasiconvex, as is an subgroup of finite index.

The concept of subgroups which are quasiconvex with respect to the set of all geodesics occurs in [Gr]. The definition here is developped in [GS1]. An interesting exercise for the reader is to show that a subgroup of a finitely generated free group is quasiconvex (with respect to the bicombing of reduced words in the free basis) if and only if is also finitely generated.

Lemma 3.1.

Let $s : G \to \mathcal{A}^\star$ be a section of the finitely generated group G. An s-quasiconvex subgroup of G is finitely generated.

Proof. We shall see that the set $\mathcal{B} = \{h \in H \mid |h|_X < 2C + 1\} = Y \cup Y^{-1}$ is a set of semigroup generators for H.

For $h \in H$, write $\sigma(h) = a_1 \dots a_n$; for each i, there is a word $\gamma_i \in F(X)$ (perhaps many such word exist) such that $a_1 \dots a_i \gamma_i$ represents an element of H, and $\ell(\gamma_i) \leq C$. Thus $h = \mu(\prod \gamma_{i-1}^{-1} a_i \gamma_i)$, where we set γ_0 to be the empty word, and h can be expressed (not necessarily uniquely) as a product of elements of $\mathcal{B}$ (of length $\ell(\sigma(w))$.

Notice that the above proof shows that the subgroup generated by a quasiconvex subset is finitely generated. However it need not be quasiconvex, as can be seen by considering $\mathbf{Z} \times \mathbf{Z}$ with the usual combing $\{x^n y^m\}$. The one element subset $\{xy\}$ is a quasiconvex subset, but does not generate a quasiconvex subgroup.

When the section is in fact a combing, more can be said:

Proposition 3.2.
If $\sigma : G \to \mathcal{A}^\star$ is a (bi-)combing, and H is a σ-quasiconvex subgroup, then H is (bi-)combable.

Proof. Take as set of semigroup generators for H the set $\mathcal{B}$ defined above. Each word $\sigma(h)$ can be expressed (not necessarily uniquely) as a product of elements of $\mathcal{B}$ of the same length; choose one of these words and call it $\sigma'(h)$.

For $b \in \mathcal{B}$, $\sigma(h)$ and $\sigma(h\mu(b))$ are uniformly $K(2K'+1)$-close in $\Gamma_X(G)$.

This means that, for each t there is a path in $\Gamma_X(G)$ labelled w_t from the point $\sigma'(h)(t)$ to the point $\sigma'(h\mu(b))(t)$ of length at most $K(2C+3)$. There are a finite number of such words w_t; let $N = \max \ell(\sigma(\mu(w)))$ where the maximum is taken over all words w of length at most $K(2C+3)$ (if the combing satisfies condition C2 then $N \leq K^2(2C+3)$). Then $\ell(\sigma'(\mu(w_t))) \leq N$, and so $\sigma'(h)$ and $\sigma'(h\mu(b))$ are uniformly N-close. ■

Thus, when σ is a combing, a σ-quasiconvex subgroup is finitely presented. Rips [R] has shown that there are finitely generated subgroups of (hyperbolic) small cancellation groups which are not finitely presentable, and also pairs of finitely presented subgroups which intersect in non-finitely generated subgroups. Thus there are subgroups which are not quasiconvex with respect to a geodesic combing. Gromov states that a subgroup of a hyperbolic group which is quasiconvex with respect to a geodesic combing is also hyperbolic (see [GS1] for a proof). Gromov also outlines an example of a finitely presented subgroup of a hyperbolic group which is itself non-hyperbolic. I do not understand the details of this example.

It is further shown in [GS1] that subgroups which are quasiconvex with respect to a (bi-)automatic structure are (bi-)automatic. The analogue of this result here is:

Proposition 3.3.
The centralizer of an element in a bicombable group is quasiconvex.

Proof. Let $x \in F(X)$, and suppose that $g\mu(x)g^{-1} = \mu(x)$, i.e. $g \in C_G(\mu(x))$. Suppose $\sigma(g) = a_1 \dots a_n$, $a_i \in X \cup X^{-1}$. There is a sequence $x = \gamma_0, \gamma_1, \dots, \gamma_n = x \in F(X)$ such that $\mu(x) = g_i\mu(\gamma_i)g_i^{-1}$, where $g_i = \mu(a_1 \dots a_i)$.

The bicombability condition ensures that we may assume that $\ell(\gamma_i) < K\ell(x) = K''$. For each word $v \in F(X)$ of length at most K'', let $\psi(v)$ be a shortest word such that $\mu(\psi(v)v\psi(v)^{-1}) = \mu(x)$, if such a word exists. For each of the words γ_i occurring above, $\psi(\gamma_i)$ exists, and there are a finite number of them.

Then for each i we have that

$$g_i\mu(\psi(\gamma_i)^{-1}x\psi(\gamma_i))g_i^{-1} = \mu(x).$$

This means that the word $\sigma(g)$ is in a Q-neighbourhood of $C_G(\mu(x))$, in the Cayley graph $\Gamma_{\mathcal{A}}(G)$, where Q is the length of the longest word $\psi(\gamma_i)$. ■

We can use the same method to show directly that in a bicombable group, the centralizer of a finite subset is also bicombable. This also follows immediately from

Proposition 3.4. *Let $\sigma : G \to \mathcal{A}^\star$ be a section for the group G, and let A, B be σ–quasiconvex subgroups. Then $A \cap B$ is a σ–quasiconvex subgroup.*

Proof. Let $g \in A \cap B$. For $i = 1, \dots, \ell(\sigma(g))$, there are elements $a(i) \in A, b(i) \in B$, and $\gamma(i), \gamma'(i) \in F(X)$ such that $\gamma(i)$ (resp. $\gamma'(i)$) is the label on a path from $\sigma(i)$ to $a(i)$ (resp. $b(i)$), and $\ell(\gamma(i)) < K_1$ (resp. $\ell(\gamma'(i)) < K_2$).

To each pair (γ, γ') of words in $F(X)$ such that $\ell(\gamma) < K_1$ and $\ell(\gamma') < K_2$, we associate a pair of elements $(\alpha(\gamma), \beta(\gamma')) \in A \times B$ (if such a pair exists) such that :

i) $\alpha \in A; \beta \in B$

ii) $\mu(\gamma)\alpha(\gamma) = \mu(\gamma')\beta(\gamma')$

iii) the sum of the lengths of $\alpha(\gamma)$ and $\beta(\gamma')$ is chosen minimal.

(If such a pair doesn't exist, then associate anything you like, we won't care in what follows.)

Now there are only a finite number of pairs (γ, γ').

For each i therefore, we have

$$\sigma(i) = a(i)\mu(\gamma(i))^{-1} = b(i)\mu(\gamma'(i)))^{-1}.$$

For each vertex $\sigma(i)$, the elements $\alpha(\gamma(i)), \beta(\gamma'(i))$ exist, and there is a bound on the sum of the lengths.

But

$$a(i)\alpha(\gamma(i)) = \sigma(i)\mu(\gamma(i))\alpha(\gamma(i)) = \sigma(i)\mu(\gamma'(i))\beta(\gamma'(i)) = b(i)\beta(\gamma(i)).$$

It follows that the path σ never strays more than a distance equal to the number of distinct pairs of elements in the product of the balls of radius K_1 and K_2 in G (bounded by the square of the number of elements in the ball of radius $\max\{K_1, K_2\}$). ■

Corollary 3.5.

A) The centralizer of a finite subset of a bicombable group is quasiconvex.

B) The centre of a bicombable group is quasiconvex and in particular is finitely generated.

The following alternative proof was shown to me by S.M. Gersten. Given a section $\sigma : G \to \mathcal{A}^\star$, say that a subset $S \subset G$ is σ-prefix closed if for all elements $s \in S$, all initial segments of $\sigma(s)$ represent elements of S. For instance the diagonal Δ is prefix closed in $G\times\cdots\times G$ for a product section $\sigma\times\ldots\sigma$. Now the intersection of a σ-quasigeodesic subgroup and a σ-prefix closed subgroup is σ-quasiconvex. Thus $\Delta \cap (C_G(x_1) \times C_G(x_2) \times \cdots \times C_G(x_n)) = C_G(x_1, x_2, \ldots, x_n)^n$.

Proposition 3.6. *If G has a short combing σ and the normal subgroup N is σ-quasiconvex, then the quotient group G/N satisfies a quadratic isoperimetric inequality.*

Proof. Again we use the notation of the proof of Proposition 2.1, where $\langle X \mid R\rangle$ is a finite presentation of G, and $\sigma : G \to F(X)$ is a short combing with constant $K = \max\{K_1, K_2\}$. We suppose in addition that R contains all words in $F(X)$ of length at most $2K + 2$ which represent the trivial element of G. Given a word $w \in F(X)$, where $w = b_1 \ldots b_n$, $b_i \in X \cup X^{-1}$, there are words $\sigma(b_1 \ldots b_j)$ which are K-fellow travellers. Following the proof of 2.1, we see that there are elements $r_i \in R$, and $p_i \in F(X)$, such that

$$w\sigma(w)^{-1} = \prod_{i=1}^{N} p_i r_i p_i^{-1} \ , \ N \leq A\ell(w)^2$$

for some constant $A \geq 0$. But the length of the word $\sigma(w)$ is at most $K\ell(w)$. The quasiconvex subgroup N has a finite set $\mathcal{B}$ of generators, which can be represented by elements of $F(X)$ of length at most $2C+1$, where C is the constant of quasiconvexity (by 3.1). Then $\langle X \mid R, \mathcal{B}\rangle$ is a finite presentation for the group G/N. With this presentation, $\sigma(w)$ is a product of $\ell(\sigma(w)) \leq K\ell(w)$ elements of $\mathcal{B}$, so that w can be expressed as a product of at most $A\ell(w)^2 + K\ell(w)$ conjugates of elements of $R \cup \mathcal{B}$, giving a quadratic isoperimetric inequality as required.

It seems hard to find examples of this phenomenum: M. Mihalik has shown that in a hyperbolic group, an infinite normal subgroup which is quasiconvex has finite index (see [MSRI, 3.8]). But we do know that the centre of a group is a normal subgroup, so that:

Corollary 3.7. *If G has a short bicombing, and $Z(G)$ denotes the centre of G, then $G/Z(G)$ satisfies a quadratic isoperimetric inequality.*

Clearly finite rank abelian groups, and finitely generated free groups give trivial examples of this phenomenon The centre of the torus knot group $\langle a, b \mid a^2 = b^3\rangle$ is generated by a^2, and killing the centre gives the free product $\mathbf{Z}_2 \star \mathbf{Z}_3$.
Question Is the quotient group bicombable? Is the same true for combable groups?

We leave as an exercise for the interested reader the proof of:

Proposition 3.8.
Let σ be a combing for G, and let H be a σ-quasiconvex subgroup.
1) For each $m \in G$, the coset Hm is a σ-quasiconvex subset.
2) If σ is a bicombing, then mH and mHm^{-1} are also σ-quasiconvex.

Section 4 Translation lengths

We follow [GS1] in this section, showing how to prove for bicombable groups the results established there for biautomatic groups.

We define the *translation number* (or stable length) of an element $g \in G$ with respect to a set of generators X to be

$$\tau_{G,X}(g) = \lim_{n\to\infty} \frac{|g^n|_X}{n}.$$

As the sequence $\{|g^n|_X\}$ is subadditive, the limit exists (see e.g. ???).

Clearly translation length of an element depends on the system of generators chosen. But it is not hard to show that

Lemma 4.1. ([Gr], [GS1, Lemma 6.1])

i) If X and Y are finite sets of generators for G, then there is a constant K such that for all $g \in G$,

$$\frac{1}{K}\tau_{G,X}(g) \leq \tau_{G,Y}(g) \leq K\tau_{G,X}(g).$$

ii) $\tau_{G,X}(g^m) = |m|\tau_{G,X}(g)$
iii) $\tau_{G,X}(hgh^{-1}) = \tau_{G,X}(g)$.

Thus for an element of a finitely generated group, the property of having non-zero translation length is independent of the finite set of generators chosen. This property is clearly inherited by finitely generated subgroups. In the reverse direction we have:

Proposition 4.2.

If H is a subgroup of G, which is quasiconvex with respect to a short combing, then for $h \in H$, $\tau_H(h) \neq 0$ iff $\tau_G(h) \neq 0$.

Proof. Choose a finite set of generators X for G, and let $\sigma : G \to F(X)$ be a short combing.

We take as finite set of generators for H the set $\mathcal{B}$ given in Lemma 2.6. Each generator $b \in \mathcal{B}$ represents an element of G which has length at most $2K+1$ with respect to X. Thus

$$|h|_X \leq (2K+1)|h|_{\mathcal{B}}$$

so that if $\tau_G(h) \neq 0$ then $\tau_H(h) \neq 0$.

As σ is a short combing, for any $g \in G$ $\ell(\sigma(g)) \leq K|g|_X$, and by proposition 2.7, there is an induced short combing $\sigma' : H \to \mathcal{B}^\star$, with constant K' say. Moreover, $\ell(\sigma(h)) = \ell(\sigma'(h))$. We thus have

$$\frac{1}{K'}|h|_{\mathcal{B}} \leq \ell(\sigma'(h)) = \ell(\sigma(h)) \leq K|h|_X.$$

The result now follows. Notice that shortness is required here to get the bounds on the lengths of representatives. ■

We immediately obtain:

Proposition 4.3. *An element of infinite order in a group with a short bicombing has non-zero translation length.*

Proof. The centralizer $C(g)$ of an element g is quasiconvex. The centre $Z(C(g))$ of this (bicombable) group is also quasiconvex. Also g lies in a finite index torsion free subgroup of the finitely generated abelian group $Z(C(g))$. With respect to a free basis, non-trivial elements of a finitely generated free abelian group have non-zero translation length. Thus applying the previous proposition three times gives the result. ■

From this we can deduce, as in [GS1], that:

Corollary 4.4.

Let G be a group with a short bicombing.

1) (see [GS1, 6.8]) *If the Baumslag-Solitar group $\langle x, y \mid xy^py^{-1} = y^q \rangle$ is a subgroup of G, then $p = \pm q$.*

2) (see [GS1, 6.9] *A virtually nilpotent subgroup of G is abelian-by-finite.*

3) (see [GS1, 6.12]) *A split extension $\mathbf{Z}^n \times_\phi \mathbf{Z}$ is a subgroup of G only if ϕ has finite order.*

4) (see [GS1, 6.15]) *A finitely generated polycyclic subgroup of G is abelian-by-finite.*

Proof. (Sketch – see [GS1] for details.)

1) This follows as, when $p \neq \pm q$, the translation length of y must be zero by 4.1. But this contradicts 4.3.

2) If a nipotent group is not abelian-by-finite, it contains elements x, y such that $[x, y] = z$ has infinite order, x and y commute with z. It follows that $[x, y]^n = [x^n, y] = [x, y^n]$ (see forinstance Rotman's booke "The theory of groups" §6.28). But then it is easy to see that the translation length of z is zero, contradicting 4.3.

3) This is because, for $x \in \mathbf{Z}^n$, $\phi^m(x)$ is conjugate to x, and hence the translation length is fixed, and so ϕ has finite order on the generators of the $\mathbf{Z}^n$ factor.

4) A polycyclic subgroup H has a finitely generated free abelian normal subgroup A. The natural map from H/A to $Aut(A)$ is a finitely generated torsion group, by part 3), and hence finite. Thus H contains a subgroup of finite index which is a direct product of the free abelain group A, and another group H_1. Now use induction on the Hirsch number, to see that H is a finite extension of an abelian group, as required.

The following application of these ideas is due to S.M. Gersten, to whom I am gateful for giving permission to reproduce his results here. What follows is taken from a letter of his dated 6th November.

Theorem 4.5. (Yau [Y])

If M is a compact Riemannian manifold of non–positive curvature, then every abelian subgroup of the fundamental group of M is finitely generated.

This result has the corollary that every solvable subgroup of the fundamental group of M has a finitely generated abelian subgroup of finite index (using structure theorems of Mal'cev) and hence is a so-called Bieberbach group. (Yau also attributes this to Gromoll and Wolf.)

To prove the theorem we need the following

Lemma 4.6. *If A is a torsion free abelian group of finite rank which is not finitely generated, then $A \setminus \{0\} \subset \mathbf{R}^n$ contains a sequence which converges to zero in $\mathbf{R}^n$, and hence the set of norms of elements of A contains 0 as a limit point.*

The following elegant proof is due to J. Stallings. If the origin is an isolated point of A, then it follows that $\mathbf{R}^n/A$ is a compact Hausdorff manifold, so its fundamental group is finitely generated.

Proof of Theorem 4.5. (Gersten)

As noted earlier, $G := \pi_1(M, \star)$ has a short bicombing. This uses the fact that geodesics in $\widetilde{M}$, the universal cover of M, beginning at the same point diverge at least as fast as Euclidean geodesics, together with the proof of the result (originally due to Svarc [Sv] – see also Milnor [M]) that the Cayley graph of G can be mapped to $\widetilde{M}$ in a quasiisometric manner. By 4.3, the translation number does not vanish on elements of infinite order.

Next one uses the fact that the Riemannian translation length function $\tau_{Riemann}$ on G is equivalent to τ_X (as defined before Proposition 2.3; this is proved in the appendix to [GS1]. Recall that the Riemannian translation length function at $g \in G$ is the translation length along a g-invariant geodesic in $\tilde{M}$, or what comes to the same thing, the length of a periodic geodesic in M in the free homotopy class of g.

Suppose now that $A < G$ is a non finitely generated abelian subgroup. Observe that A is torsion free and of rank at most the dimension of M; this follows since M is a space of type $K(G, 1)$. If we consider τ_X restricted to A and use Lemma 4.6, we see there is a sequence of elements $a_n \in A \setminus \{0\}$ with $\tau_X(a_n) \to 0$ as $n \to \infty$. It follows that $\tau_{Riemann}(a_n) \to 0$ also.

But this means there is a sequence of non trivial periodic geodesics in M with lengths tending to zero. This is absurd, since there is a positive lower bound for the lengths of such geodesics, the systole of M. This completes the proof of the theorem.

Remark. (Gersten) One would like to be able to show that $\tau_X(G)$ is discrete at 0 without recourse to Riemannian geometry, assuming that G is bicombable. If this were the case, it would follow that every torsion free finite rank abelian subgroup of a bicombable group was a lattice ($\cong \mathbf{Z}^n$).

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