# The subgroups of direct products of surface groups

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**Abstract.** A subgroup of a product of $n$ surface groups is of type $FP_n$ if and only if it contains a subgroup of finite index that is itself a product of (at most $n$) surface groups.

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*For John Stallings on his 65th birthday.*

By a *surface group* we mean the fundamental group of a connected 2-manifold. Such a group is either free (of finite or countably infinite rank) or else has a subgroup of index at most two with a presentation of the form $\mathcal{P}_g = \langle a_1, b_1, \ldots, a_g, b_g \mid [a_1, b_1] \ldots [a_g, b_g] \rangle$.

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In this article we shall calculate the finiteness properties of all subgroups of (finite) direct products of surface groups. Uncountably many non–isomorphic groups arise as such subgroups [4], and a celebrated theorem of Stallings [14] and Bieri [1] shows that the full range of possible finiteness properties is to be found amongst these examples.

In contrast to this diversity, we shall prove that the only subgroups that enjoy the fullest degree of homological finiteness are the most obvious ones:

THEOREM A. *Let $G$ be a subgroup of a direct product of $n$ surface groups. If $G$ is of type $FP_n$, then $G$ is virtually a direct product of at most $n$ finitely generated surface groups.*

In the case of products of free groups, this theorem generalizes results of Grunewald [6], Meinert [11], Johnson [8], [9], and Baumslag and Roseblade [4]. In this last article Baumslag and Roseblade showed that a finitely presented subgroup of a direct product of two free groups is virtually a direct product of free groups (see also [2], [13]). Trying to better understand and generalize their result was the starting point of this investigation.

Note that if $G$ is a subgroup of a direct product $A \times B$ such that $G \cap A$ is trivial, then $G$ is isomorphic (via projection) to a subgroup of $B$. (Here and elsewhere we abuse notation by using $A$ to denote the subgroup $A \times 1$ of $A \times B$.) Thus Theorem A is an easy consequence of the following generalization of results of [4]:

THEOREM B. *Let $F_1, \ldots, F_n$ be surface groups (not necessarily finitely generated). Let $G$ be a subgroup of their direct product $F_1 \times \cdots \times F_n$ and assume that each $L_i = G \cap F_i$ is non-trivial for $i = 1, \ldots, n$.*

*If we arrange the notation so that $L_1, \ldots, L_r$ are not finitely generated and $L_{r+1}, \ldots, L_n$ are finitely generated, then $G$ contains a subgroup of finite index $G_0$ such that:*

1. *$G_0 = B \times {F_{r+1}}' \times \ldots \times {F_n}'$, where each $F_i'$ is a finitely generated subgroup of $F_i$ and $B$ is a subgroup of $F_1 \times \cdots F_r$,*
2. *if $r \geq 1$, then $H_r(B, \mathbb{Z})$ is not finitely generated.*

*In particular, if precisely $r \geq 1$ of the $L_i$ are not finitely generated, then $G$ is not of type $FP_r$.*

This result settles questions raised in [2], [5], [8] and [11]. Notice that the theorem immediately generalizes to products of finite extensions of surface groups.

## 1. Ingredients Needed in the Proof

The ingredients in the proof are surface group analogues of those used by Baumslag and Roseblade [4] for free groups together with induction. In this section we establish the required facts for surface groups.

### 1.1. Spectral sequences.

The following spectral sequence observation from the homology of groups enables us to carry out an inductive argument.

LEMMA 1.1. *Let $Q$ be a group of cohomological dimension at most 2 and consider a short exact sequence $1 \to N \to E \to Q \to 1$. If $H_1(Q, H_k(N))$ is not finitely generated for some $k \geq 0$ then $H_{k+1}(E)$ is not finitely generated.*

*Proof.* Consider the Lyndon–Hochschild–Serre spectral sequence with $E^2_{p,q} = H_p(Q, H_q(N))$ (see p.171 [3] for example). Since $Q$ has dimension at most 2, the only non-zero terms in the $E^2$ page of the spectral sequence are in columns 0,1 and 2. In particular there are no non-zero derivatives involving the terms in column 1, and therefore $H_1(Q, H_k(N)) = E^2_{1,k} = E^\infty_{1,k}$ is a section (= quotient of a subgroup) of $H_{k+1}(E)$.

### 1.2. Finding primitive elements.

Recall that an element $a$ in a free group $F$ is said to be *primitive* if $a$ is part of a free basis for $F$. We need the following easy observation:

LEMMA 1.2. *If $a$ is primitive in $F$ and $L \leq F$ is any subgroup containing $a$, then $a$ is also primitive in $L$.*

*Proof.* Since $a$ is primitive, we can realize $F$ as the fundamental group of a wedge of simple loops joined at a base point where one of the loops $\alpha$ represents $a$. Since $a \in L$, in the covering space corresponding to $L$, the loop $\alpha$ lifts to a simple loop $\tilde{\alpha}$ at the base point which represents $a$. Then the usual method of finding a basis for $L$ includes $a$ in the basis. (Alternatively, this lemma can be deduced from the Kurosh Subgroup Theorem.)

According to a theorem of M. Hall [7](see also [15]), for any non–trivial element $b \in F$, the cyclic subgroup $\langle b \rangle$ is a free factor of a subgroup $\hat{F}$ of finite index in $F$. Thus $b$ is primitive in $\hat{F}$. We record this as

LEMMA 1.3. *If $b$ is a non–trivial element in a free group $F$, then $b$ is a primitive element in some subgroup of finite index in $F$.*

Similarly, if $S$ is a closed orientable surface, an element $a \in \pi_1(S)$ is said to be *primitive* if the free homotopy class of $a$ contains a non-separating simple closed curve on $S$. Such an $a$ can be chosen to be the generator $a_1$ in the standard presentation $\pi_1(S) = \langle a_1, b_1, \ldots, a_g, b_g \mid [a_1, b_1] \ldots [a_g, b_g] = 1 \rangle$.

We need the analogues of the previous two lemmas for closed surface groups.

LEMMA 1.4. *If $S$ is a closed orientable surface and $a_1$ is primitive in $\pi_1(S)$ and $L \leq \pi_1(S)$ is any subgroup containing $a_1$, then $a_1$ is also primitive in $L$.*

*Proof.* We can assume the notation is chosen so that the primitive element is $a_1$ in the standard presentation. Suppose $a_1$ is contained in the subgroup $L$. If $L$ has finite index, then the simple loop representing $a_1$ lifts in the corresponding covering space to a simple non-separating loop which again represents a primitive element.

If $L$ has infinite index, then $L$ is free and we need to show $a_1$ is part of a basis. The surface $S$ has a cell decomposition consisting of a single 2-cell with $4g$ boundary edges which get identified according to the defining relation of $\mathcal{P}_g$. $S$ can then be triangulated by joining each boundary vertex and the midpoint of each boundary edge to a single central point in the 2-cell. In this triangulation, there are two triangles, say $\Delta_1$ and $\Delta_2$, with all three vertices in common and sharing a common edge so that $a_1$ is represented by a loop consisting of two edges in the boundary of $\Delta_1 \cup \Delta_2$

Let $\tilde{S}$ be the triangulated covering space corresponding to $L$. In the textbooks by Massey [10, pages 199–200] and Stillwell [16, pages 142–144] are proofs that $\pi_1(\tilde{S})$ is free. They construct inductively a basis for the fundamental group which is the union of bases for expanding finite subcomplexes which deformation retract onto a subgraph. $\Delta_1 \cup \Delta_2$ lifts homeomorphically to the union of two triangles $\widetilde{\Delta}_1 \cup \widetilde{\Delta}_2$ and it is clear we can start the construction with these so that the lift of the loop representing $a_1$ is the first basis element. This proves the lemma.

LEMMA 1.5. *Let $S$ be a closed orientable surface of genus at least 2 and let $a \in \Gamma = \pi_1(S)$ be a non–trivial element which is not a proper power. There is a finite index subgroup of $\Gamma$ in which $a$ is primitive.*

*Proof.* This result is an application of the fact that surface groups are LERF (see [12]). Fix a metric of constant curvature on $S$ and let

$\alpha$ be a closed geodesic on $S$ representing the free homotopy class of $a$. If $\alpha$ is not simple, then $\alpha$ contains a proper embedded subloop $\alpha'$. Since the length of the closed geodesic homotopic to $\alpha'$ has length less than $\alpha$, and $\alpha$ is not a proper power, the conjugacy class represented by $\alpha'$ does not intersect the cyclic subgroup generated by $a$. Since $\pi_1 S$ is LERF, we may pass to a subgroup of finite index $H \subset \pi_1 S$ that contains $\langle a \rangle$ but has empty intersection with the conjugacy class $[\alpha']$.

The loop $\alpha$ lifts to a loop in the finite sheeted covering $\hat{S} \to S$ corresponding to $H$ but $\alpha'$ does not. Thus the lift of $\alpha$ has fewer self-intersection points (counted with multiplicity) than $\alpha$. By repeating this argument a finite number of times (with $\pi_1 \hat{S}$ in place of $\pi_1 S$) we obtain a finite sheeted covering in which $a$ is represented by a simple closed loop. If this closed loop separates, then $\pi_1 \hat{S}$ is a free product with amalgamation $A *_{\langle a \rangle} B$. We can take a further 2-sheeted covering corresponding to any subgroup of index 2 not containing $A$ or $B$. The element $a$ belongs to such a subgroup (since it is null–homologous) and in the corresponding covering is represented by a simple non–separating loop.

## 1.3. Free differential calculus.

We want to apply the spectral sequence observation of 1.1 when there is a primitive element which acts trivially. The following is from [4]:

LEMMA 1.6. *Let $F$ be a free group and suppose $M$ is a right $F$-module. If a primitive element of $F$ acts trivially on $M$, then the homology group $H_1(F, M)$ contains an isomorphic copy of $M$.*

*Proof.* We may suppose that $a_1, a_2, \ldots$ is a basis for $F$ and that $a_1$ acts trivially on $M$. Recall that $H_1(F, M)$ is the kernel of the map $\bigoplus M \to M$ defined by

$$(m_1, m_2, \ldots) \mapsto m_1(1 - a_1) + m_2(1 - a_2) + \cdots$$

Clearly the first summand $M$ lies in the kernel since $a_1$ acts trivially.

We need a similar result for closed orientable surface groups. To calculate the homology of such a group one uses the free differential calculus (Fox derivatives) to write down the second boundary map in chain complexes of modules over the group ring of a closed surface (see [3], pages 45,46)

LEMMA 1.7. *Let $G = \langle a_1, b_1, \ldots, b_g \mid [a_1, b_1] \ldots [a_g, b_g] \rangle$ be the group of a closed orientable surface, and suppose that $M$ is a (right) $\mathbb{Z}G$-module on which $a_1$ acts trivially (so that $M(1 - a_1) = 0$). If $M$ has infinite $\mathbb{Z}$-rank, then so does $H_1(G, M)$.*

*Proof.* The presentation

$$G = \langle a_1, b_1, \ldots, a_g, b_g \mid a_1 b_1 a_1^{-1} b_1^{-1} \ldots a_g b_g a_g^{-1} b_g^{-1} \rangle$$

is aspherical, and gives rise to a resolution

$$\mathcal{F} : 0 \to \mathbb{Z}G \xrightarrow{d_2} \mathbb{Z}G^{2g} \xrightarrow{d_1} \mathbb{Z}G \xrightarrow{\epsilon} \mathbb{Z} \to 0$$

of free (left) $\mathbb{Z}G$-modules. Here $\epsilon$ is the augmentation map, $d_1$ is given by

$$d_1(m_1, \ldots, m_{2g}) = m_1(1 - a_1) + m_2(1 - b_1) + \ldots + m_{2g}(1 - b_g),$$

and $d_2$ by the Fox-derivatives of the relator

$$R \equiv a_1 b_1 a_1^{-1} b_1^{-1} \ldots a_g b_g a_g^{-1} b_g^{-1} :$$

$$d_2(m) = (m\frac{\partial R}{\partial a_1}, m\frac{\partial R}{\partial b_1}, \ldots, m\frac{\partial R}{\partial b_g}).$$

Recall that, for a basis element $x$ and word $R$ in a free group, the Fox derivative $\frac{\partial R}{\partial x}$ is determined by the recursive rules

$$\frac{\partial x}{\partial x} = 1; \quad \frac{\partial RS}{\partial x} = \frac{\partial R}{\partial x} + R\frac{\partial S}{\partial x}.$$

In particular it follows that $\frac{\partial 1}{\partial x} = 0$ and $\frac{\partial x^{-1}}{\partial x} = -x^{-1}$. In our case, each basis element appears exactly twice in the relator $R$, and we can express the Fox derivative as a sum of two terms:

$$\frac{\partial R}{\partial a_i} = (a_1 b_1 a_1^{-1} b_1^{-1} \ldots a_{i-1} b_{i-1} a_{i-1}^{-1} b_{i-1}^{-1})(1 - a_i b_i a_i^{-1}),$$

$$\frac{\partial R}{\partial b_i} = (a_1 b_1 a_1^{-1} b_1^{-1} \ldots a_{i-1} b_{i-1} a_{i-1}^{-1} b_{i-1}^{-1})(a_i - a_i b_i a_i^{-1} b_i^{-1}).$$

Since $M$ is a right $\mathbb{Z}G$-module, we may calculate $H_*(G, M)$ as the homology of the chain complex (of $\mathbb{Z}$-modules)

$$M \otimes_{\mathbb{Z}G} \mathcal{F} : 0 \to M \xrightarrow{d_2} M^{2g} \xrightarrow{d_1} M \to 0.$$

Here, again, $d_2$ is given by the Fox derivatives:

$$d_2(m) = (m\frac{\partial R}{\partial a_1}, m\frac{\partial R}{\partial b_1}, \ldots, m\frac{\partial R}{\partial b_g}).$$

Now the fact that $a_1$ acts trivially allows us to simplify the expressions for some of the Fox derivatives. In particular, note that $a_1b_1a_1^{-1}b_1^{-1}$ acts trivially on $M$, since $a_1$ does. Hence the expression for $d_2(m)$ can be written as

$$d_2(m) = (m(1-b_1), 0, m(1-a_2b_2a_2^{-1}), m\frac{\partial R}{\partial b_2}, \dots, m\frac{\partial R}{\partial b_g}).$$

Also note that $\frac{\partial R}{\partial b_2}$ acts on $M$ by

$$m\frac{\partial R}{\partial b_2} = ma_2 - ma_2b_2a_2^{-1}b_2^{-1}.$$

We will need the following two subgroups of the module $M$:

$$A = \{m \in M : m\frac{\partial R}{\partial b_1} = m\frac{\partial R}{\partial a_2} = \ldots = m\frac{\partial R}{\partial b_g} = 0\}$$

and

$$B = \{m \in M : m\frac{\partial R}{\partial a_1} = 0\} = \{m \in M : m(1-b_1) = 0\}.$$

Observe that $B_1 = 0 \oplus B \oplus 0 \oplus \cdots \oplus 0$ lies in the kernel of $d_1$ and $B_1 \cap d_2(M) = 0$ by the above formula for $d_2(m)$. Hence $B$ is isomorphic to a subgroup of $H_1(G,M)$. Moreover if $d_2(m) \in 0 \oplus M \oplus \cdots \oplus M$ then $m(1-b_1) = 0$ so that $m \in B$. Thus

$$d_2(B) = d_2(M) \cap (0 \oplus M \oplus \cdots \oplus M).$$

Next consider

$$A_1 = 0 \oplus 0 \oplus Aa_2b_2a_2^{-1} \oplus 0 \oplus \ldots \oplus 0.$$

This group is contained in the kernel of $d_1$, since

$$ma_2b_2a_2^{-1}(1-a_2) = -m\frac{\partial R}{\partial b_2}b_2 = 0$$

for all $m \in A$ by definition of $A$. Hence $\frac{A_1}{A_1 \cap d_2(M)} = \frac{A_1}{A_1 \cap d_2(B)}$ is isomorphic to a subgroup of $H_1(G,M)$.

Suppose now on the contrary that $H_1(G,M)$ has finite $\mathbb{Z}$-rank. Then so do $B$ and $\frac{A_1}{A_1 \cap d_2(B)}$ and hence also $A_1 \cong A$.

Since $M(1-a_1) = 0$, the subgroup $M_1 := M \oplus 0 \oplus \ldots \oplus 0$ is contained in the kernel of $d_1$. Thus $\frac{M_1}{M_1 \cap d_2(M)}$ is contained in $H_1(G,M)$ and hence has finite $\mathbb{Z}$-rank. Note also that

$$M_1 \cap d_2(M) \cong A\frac{\partial R}{\partial a_1}$$

and hence
$$\frac{M_1}{M_1 \cap d_2(M)} \cong \frac{M}{A\frac{\partial R}{\partial a_1}}.$$
But $A\frac{\partial R}{\partial a_1}$ has finite $\mathbb{Z}$-rank since $A$ does. Thus $M$ has finite $\mathbb{Z}$-rank which is a contradiction. This completes the proof.

## 2. Proof of Theorem B

Since the conclusion of the theorem allows us to pass to a subgroup of finite index, we may immediately replace the given $G$ by its intersection with the product of the orientation-preserving subgroups of any closed surface factors $F_i$. Also, if any of the surface groups is $\mathbb{Z} \times \mathbb{Z}$ corresponding to a 2-torus, we choose to regard this as the product of two surface groups with $\mathbb{Z}$ (free of rank 1) fundamental group. In other words, we may assume that all of the $F_i$ are either free or else have a presentation of the form of $\mathcal{P}_g$ with $g \geq 2$.

Let $L_i$ be as in the statement of the theorem. Let $\rho_i : G \to F_i$ be the projection of $G$ to $F_i$. Observe that $P_i = \rho_i(G)$ is a surface group and $L_i = G \cap \rho_i(G)$ is normal in $P_i$.

In particular, if $L_i$ is finitely generated, then it must be of finite index in $P_i$. In this case, the subgroup $G' = \rho_i^{-1}(L_i)$ has finite index in $G$ and $\rho_i(G') = L_i$. Thus $L_i$ splits as a direct factor of $G'$.

Applying this to each of those factors $F_i$ with $i > r$ for which $L_i$ is finitely generated produces a subgroup $G_1 = \cap_{i=r}^{n} \rho_i^{-1}(L_i)$ of finite index in $G$ with $G_1 = B_1 \times L_{r+1} \times \cdots \times L_n$ where $B_1 = G \cap (F_1 \times \cdots \times F_r)$.

We now turn our attention to the $L_i$ which are not finitely generated. Since each of $L_1, \ldots, L_r$ is non-trivial, there is some $1 \neq c_i \in L_i = G \cap P_i$. By Lemmas 1.3 and 1.5, each $c_i$ is primitive in a subgroup $\hat{P}_i$ of finite index in $P_i$. Let $G_0 = G_1 \cap \rho_1^{-1}(\hat{P}_1) \cap \cdots \cap \rho_r^{-1}(\hat{P}_r)$. Then $B = G_0 \cap B_1$ has finite index in $B_1$, each $c_i \in G_0 \cap F_i$ and each $c_i$ is primitive in $\rho_i(G_0)$ for $i = 1, \ldots, r$.

Of course $G_0$ has finite index in $G$ and $G_0 = B \times L_{r+1} \times \cdots \times L_n$. Theorem B is now an immediate consequence of the following:

LEMMA 2.1. *Let $B$ be a subgroup of a direct product of $r$ surface groups $F_1 \times \cdots \times F_r$ where each $F_i$ is free or the group of a closed orientable surface of genus at least 2. Let $\rho_i$ denote the projection from $B$ to $F_i$ and put $P_i = \rho_i(B)$. Suppose the following:*

1. *each of the intersections $L_i = B \cap F_i$ is not finitely generated; and*
2. *each $L_i$ contains an element that is primitive in $P_i$.*

*Then $H_r(B, \mathbb{Z})$ is not finitely generated.*

*Proof.* We shall prove the lemma by induction on $r$. The case $r = 1$ is trivial. In the inductive step we consider the projection of $B$ onto the last factor:

$$1 \to N \to B \to P_r \to 1.$$

$N$ is the intersection of $B$ with $F_1 \times \ldots \times F_{r-1}$ and its intersections with the factors $F_i$ are those of $B$ (for $i = 1, \ldots, r-1$). In particular, each $L_i$ still contains a primitive element of $\rho_i(N)$. Thus, by induction, we may assume that $H_{r-1}(N, \mathbb{Z})$ is not finitely generated.

Now $M = H_{r-1}(N, \mathbb{Z})$ can be viewed as a right $P_r$-module coming from the conjugation action of $B$ on $N$. By hypothesis $P_r$ contains a primitive element which lies in $L_r$ and hence acts trivially on $N$. Thus by Lemma 1.6 or 1.7, $H_1(P_r, M)$ is not finitely generated. That is, $H_1(P_r, H_{r-1}(N, \mathbb{Z}))$ is not finitely generated. Hence by Lemma 1.1 $H_r(B, \mathbb{Z})$ is not finitely generated.

This completes the proof of the Lemma and hence Theorem B.

We note that an argument similar to the above also establishes the following general fact:

PROPOSITION 2.2. *Let $1 \to N \to G \to F \to 1$ be a short exact sequence of groups such that*

1. *$F$ is a surface group,*
2. *$C_G(N) \not\subset N$, and*
3. *the k-th integral homology $H_k(N, \mathbb{Z})$ is not finitely generated.*

*Then $G$ has a finite index subgroup $G_0$ whose (k+1)-st integral homology $H_{k+1}(G_0, \mathbb{Z})$ is not finitely generated.*

*Proof.* Since $C_G(N) \not\subset N$, the quotient $F$ contains a non-trivial element $c$ which acts trivially on $H_*(N, \mathbb{Z})$. Since $F$ is a surface group, $c$ is primitive in some subgroup $F_0$ of finite index in $F$. Let $G_0$ be the preimage of $F_0$ in $G$ which also has finite index. By Lemma 1.6 or 1.7, $H_1(G_0, H_k(N, \mathbb{Z}))$ is not finitely generated. Hence by Lemma 1.1, $H_{k+1}(G_0, \mathbb{Z})$ is not finitely generated.

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