FINITELY PRESENTED SUBGROUPS OF A PRODUCT OF TWO FREE GROUPS

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Abstract. We use van Kampen diagram techniques to give a new geometric proof of Baumslag and Roseblade's result that a non–free finitely presented subgroup of a direct product of two free groups is virtually a product of subgroups of the factors.

§ Introduction

The usefulness of geometric methods in combinatorial group theory is now well–established. We present here an idea concerning diagrams and subgroups, and use it to give an elementary proof of G. Baumslag and J. E. Roseblade's result:

Theorem [1, Thm. 2]
Let H be a finitely presentable subgroup of a direct product $A \times B$ of two free groups A, B. If H is not free, then H contains a subgroup $A' \times B'$ of finite index, where A' (resp. B') is a subgroup of A (resp. B).

To see that the 'finitely presentable' part of the statement is necessary, note that K. A. Mihailova [4] and C. F. Miller III [5] have shown that the direct product of two free groups contains finitely generated subgroups with many unsolvable properties. For instance, if $\langle X; R\rangle$ is a finite presentation of a group with unsolvable word problem, the subgroup of $F(X) \times F(X)$ generated by $\{(x, x), (1, r); x \in X, r \in R\}$ has unsolvable membership problem. Note also that there are continuously many non–isomorphic subgroups of the direct product of two non–abelian free groups [1, Thm.1].

To see that the 'finite index' part of the statement is necessary, consider the following simple example, where F denotes the free group of rank 2. If $\phi : F \to \mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$ is an epimorphism, then the kernel K of the map $F \times F \to \mathbb{Z}_n$ defined by $(a, b) \to \phi(a)\phi(b)$ is finitely generated, but is not a direct product of subgroups of the factors, as $K \cap (F \times \{1\})$ has index n in $F \times \{1\}$, while the finitely presentable group K has index n in $F \times F$.

The proof of the theorem given by Baumslag and Roseblade uses group homology and spectral sequences. M. Bridson and D. Wise [2]

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have used the theory of CAT(0) spaces (more precisely $\mathcal{V}H$–complexes) to produce a new geometric proof of the theorem, which has some intricate aspects but is basically elementary.

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§ Diagrams, Cayley graphs and the proofs

We begin with a brief description of van Kampen diagrams (see Lyndon and Schupp's book [3] for a more complete description).

Let $\mathcal{P} = \langle Z_{\mathcal{P}}; R\rangle$ denote a finite presentation of the group G, and let $\mathcal{Q} = \langle Z_{\mathcal{Q}}; S\rangle$ denote a finite presentation of a subgroup H of G. The Cayley graph of G with respect to the generating set $Z_{\mathcal{P}}$ is denoted $\Gamma_{Z_{\mathcal{P}}}(G)$, and we regard it as a metric space in the usual way. We regard H as a subset of (the vertices of) $\Gamma_{Z_{\mathcal{P}}}(G)$. A *relation* over $\mathcal{P}$ is a word $w \in F(Z_{\mathcal{P}})$ (or in the free monoid $(Z_{\mathcal{P}} \cup Z_{\mathcal{P}}^{-1})^*$) that represents the identity element in G. Expressing such a relation as a product of conjugates of relators from R gives rise to a reduced, based, van Kampen diagram over $\mathcal{P}$, or $\mathcal{P}$–diagram, $D_{\mathcal{P}}(w)$ (see for instance [3]). This diagram can be viewed as a compact, connected, planar 2–complex, where the edges are oriented and labelled by generators (from $Z_{\mathcal{P}}$), and the boundaries of the 2–cells are labelled by the relators (from R), when read from an appropriate initial vertex, with the appropriate orientation. The base point of $D = D_{\mathcal{P}}(w)$ is a vertex on the boundary from which the word w is read around the outside boundary (i.e. the boundary of the complement in $\mathbb{R}^2$). The diagram is *unreduced* if there is an (oriented) edge e shared by two 2–cells such that the labels on the boundaries of the two 2–cells are identical, when read from the initial point of the edge e, with the orientations induced by e. An unreduced diagram can be reduced by cancelling such pairs of faces — first removing the two faces and their common edge, and then identifying the two halves of the boundary of the removed faces. The choice of base point induces a natural label–preserving morphism of graphs $\Phi : D^{(1)} \to \Gamma_{Z_{\mathcal{P}}}(G)$ of the 1–skeleton of the based van Kampen diagram D to the Cayley graph $\Gamma_{Z_{\mathcal{P}}}(G)$ (based at the identity vertex).

Via the inclusion map $i : H \hookrightarrow G$, for each H–generator $z \in Z_{\mathcal{Q}}$, we choose a reduced word $p(z) \in F(Z_P)$ such that $i(z) =_G p(z)$. The map p extends to a homomorphism from the free monoid $(Z_{\mathcal{Q}} \cup Z_{\mathcal{Q}}^{-1})^*$ to $(Z_{\mathcal{P}} \cup Z_{\mathcal{P}}^{-1})^*$. We shall usually suppress reference to the inclusion map i. If w is a relation over $\mathcal{Q}$, then the reduced form of $p(w)$ bounds a reduced $\mathcal{P}$ diagram $D_{\mathcal{P}}(p(w))$ (in general this diagram is not unique).

We begin with a general result about finitely presented subgroups of finitely presented groups.

Lemma 1. *Let $\mathcal{P}, \mathcal{Q}, G, H, p$ be as above. There is a constant K, such that, for any relation w over $\mathcal{Q}$, there is a reduced diagram $D_{\mathcal{P}}(p(w))$ with the property that $\Phi(D_{\mathcal{P}}(p(w))^{(0)})$, the image in $\Gamma_{Z_{\mathcal{P}}}(G)$ of its 0–skeleton, lies in a K–neighbourhood of H.*

Proof. A diagram $D = D_{\mathcal{Q}}(w)$ for w over the presentation $\mathcal{Q} = \langle Z_{\mathcal{Q}}; S\rangle$ (the subgroup H) gives rise to a diagram D' for $p(w)$ over $\mathcal{P} = \langle Z_{\mathcal{P}}; R\rangle$ (the group G) as follows. Each edge labelled z_i in D is relabelled $p(z_i)$ (after the appropriate subdivision). For each relator $s \in S$, choose a diagram $D_{\mathcal{P}}(p(s))$ over $\mathcal{P}$ for the image $p(s)$. At each face of D which was labelled s, insert a copy of $D_{\mathcal{P}}(p(s))$. After some collapsing of edges, we thus obtain a diagram D' for $p(w)$ over $\mathcal{P}$ (which may be unreduced). Let k_s be the maximal distance of a vertex in $D_{\mathcal{P}}(p(s))$ from the boundary of $D_{\mathcal{P}}(p(s))$. As the vertices of D lie in H, the vertices of D' (after the map Φ) lie in a $(\max_{s\in S}\{k_s\} + \max_{z\in Z_{\mathcal{Q}}}\{\ell(p(z))\})$ neighbourhood of H in $\Gamma_{Z_{\mathcal{P}}}(G)$ ($\ell(w)$ denotes the length of the word w).

Let D'' be a diagram obtained from D' by reduction. Reduction of a diagram cancels faces, but $\Phi(D''^{(0)}) \subset \Phi(D'^{(0)})$. A finite number of reductions gives a diagram $D_{\mathcal{P}}(p(w))$ with the required property. □

Now we restrict to the case of the theorem. Let X (resp. Y) be a set of free generators for the free group A (resp. B). Let $G = A \times B$, and let $\mathcal{P}$ denote its finite presentation $\langle X, Y; [x_i, y_j],\ x_i \in X,\ y_j \in Y\rangle$. As only finitely generated subgroups H are considered here, it suffices to consider the case when X and Y are finite. We shall use extensively the following observation:

Lemma 2. *Let H be a subgroup of a direct product of two free groups $G = A \times B$. If H is not free, then H contains non–trivial elements $a \in A \times \{1\}$ and $b \in \{1\} \times B$.*

Proof. Let p_A, p_B be the projections onto the first and second factors. If the restriction of p_A to H is injective, then H is isomorphic to a subgroup of a free group, and hence free. Otherwise H contains a non–trivial element of the kernel, i.e. of $\{1\} \times B$. Similarly if H is not free, then it contains a non-trivial element of $A \times \{1\}$. □

We shall use these elements to manufacture for each $h \in H$ diagrams that contain arcs labelled by $p_A(h)p_B(h)$ onto each factor. To do this we consider paths in van Kampen diagrams in more detail.

Let $\mathcal{Q} = \langle Z; S\rangle$ be a finite presentation of the subgroup H of $G = A \times B$. For each $z_j \in Z$ there are unique reduced words $u_j \in F(X), v_j \in F(Y)$ such that $z_j =_G u_j v_j$. Define $p(z_j) = u_j v_j$, and for $w \in F(Z)$, the word $p(w)$ is obtained by replacing each generator $z_i^{\pm 1}$ in w by $(u_i v_i)^{\pm 1}$.

A path in the 1–skeleton of a (reduced or unreduced) $\mathcal{P}$–diagram is an X*–arc* (respectively a Y*–arc*) if the edges contained in the arc are all labelled by letters in X (resp Y). The arc is *reduced* if the word labelling the arc is reduced.

We first establish a result for diagrams which are topological discs. The general case follows immediately.

Lemma 3. *Let D be a reduced diagram over $\mathcal{P}$ that is a topological disc. Every X–edge in the interior* $\operatorname{Int} D$ *lies in a reduced, properly embedded X–arc.*

Proof. Let e be an oriented X–edge in $\operatorname{Int} D$. Let $\mathcal{F}$ be the face of D containing the oriented X–edge e in its boundary and lying to the right of e. Let ${q_1}', q_1$ be the initial and final vertices of e.

Let ${f_1}' e f_1$ be the subsequence of three edges of $\partial\mathcal{F}$ containing, and oriented by, the edge e. The edges ${f_1}', f_1$ are Y–edges. Let $\mathcal{F}_1$ (respectively ${\mathcal{F}_1}'$) be the other face of D containing the edge f_1 (resp. ${f_1}'$), if q_1 (resp. ${q_1}'$) is not a boundary vertex.

The region $\mathcal{F}_1$ contains an X–edge e_1 with initial vertex q_1, and ${f_1}^{-1} e_1 f_2$ is the sequence of 3 edges of F_1 containing e_1. Let x be the label on the oriented edge e, and y be the label on the oriented edge f_1. The label on F, with the orientation induced by e, is $y^{-1}xyx^{-1}$, and if x' is the label on the oriented edge e_1, then the label on F_1, with the orientation induced by e_1, is $y^{-1}x'yx'^{-1}$. If the label x' on e_1 were the inverse of the label x on e, the two faces F and F_1 would cancel, contradicting the assumption that the diagram is reduced.

Let q_2 be the endpoint of e_1. Let $\mathcal{F}_2$ be the other face containing the edge f_2 (if q_2 is not on ∂D), and let e_2 be the X–edge of $\mathcal{F}_2$ meeting the vertex q_2. As before, the label on e_2 is not the inverse of the label on e_1. Continuing in this way, a sequence of edges $e, e_1, e_2, \dots, e_n$ is constructed, terminating at the vertex $q_n \in \partial D$. Similarly a sequence ${e_m}', \dots {e_1}', e$ of edges is constructed, starting with the vertex ${q_m}' \in \partial D$.

The path is labelled by a reduced word, by the remark concerning the labelling of successive X–edges, and the sequence of vertices ${q_m}', \dots {q_1}',\ q_1, q_2, \dots q_n$ contains no repetitions (else there is a closed X–path bounding a diagram over $\mathcal{P}$, which is impossible as A is a free group and the label is a reduced word).

Thus the path is indeed a reduced embedded X–arc. □

The main theorem follows immediately from the following proposition.

Proposition 4. *Let H be a finitely presentable subgroup of a direct product $F(X) \times F(Y)$ of two free groups.*

If H is not free, then there is a finite set V of words in $F(X, Y)$, and finite sets of words $\{a_i\} \subset F(X)$, $\{b_j\} \subset F(Y)$, all representing elements of H, such that every element of $h \in H$ can be expressed as a product $h'h''$, where h' is a product of the elements $\{a_i, b_j\}$, and $h'' \in V$.

Proof Let a and b be non–trivial elements provided by Lemma 2, and choose a finite presentation $\langle Z; S\rangle$ for H with $a, b \in Z$. The property to be established is invariant under conjugation, so assume that $p(a)$ and $p(b)$ are cyclically reduced words in $F(X) \cup F(Y)$.

Let h be a reduced word in $F(Z)$, and let $c \in F(X)$, $d \in F(Y)$ be the freely reduced words such that $h =_G cd$. (In general $p(h)$ is not the word cd.) If there is a bound on the lengths of c and d over all h, then H is finite, and therefore trivial, as G is torsion–free.

Let K denote the constant provided in Lemma 1, and let M denote the number of elements of G of length at most K. Let V consist of all elements $g \in H$ such that $g =_G cd$ with $\max\{\ell_X(c), \ell_Y(d)\} \leq 2M$.

Now suppose that $h \in H$ is such that $\ell(c) > M$. We prove that there is an element $z \in (A \times \{1\}) \cap H$, with $\ell(z) \leq 2M$, so that $h =_G cd =_G z\bar{h}$, $\bar{h} =_G c'd$, and $\ell(c') < \ell(c)$. The result then follows from this and the analogous result for d.

Choose a large integer N so that $p(a)^N$ and $p(b)^N$ are longer than $p(h)$. The product $w = b^N h a^{2N} h^{-1} b^{-2N} h a^{-2N} h^{-1} b^N$ is a relation in H, and Lemma 1 provides a reduced van Kampen diagram $D = D_{\mathcal{P}}(p(w))$ such that $\Phi(D)$ lies K–close to H in $\Gamma_{X,Y}(G)$.

Let q_0 be the base vertex of the diagram D. As no proper initial subword of $p(w)$ is trivial in G, q_0 is the endpoint of an X–edge in the interior of D. By Lemma 3, q_0 is the initial vertex of an embedded, reduced X–arc α in D. The only proper initial segment of $p(w)$ that is equal in G to an X–word is $p(b)^N p(h) p(a)^{2N} p(h)^{-1} p(b)^{-N}$, by the choice of N. Thus the label on α in the group $G = A \times B$ represents the element $ha^{2N}h^{-1} =_G ca^{2N}c^{-1}$. If the last letter of (the reduced word) c cancels with the first letter of $p(a)$, then replace a by a^{-1} in the above. The word $cp(a)$ is reduced as written, and the word c is an initial segment of $cp(a)^{2N}c^{-1}$, of length $\ell(c) \geq M + 1$.

Let $q_0, q_1, \ldots, q_{M+1}$ be the first $M + 2$ vertices of D of the path α. For each q_i, there is a path γ_i in $\Gamma_{X,Y}(G)$ of length at most K, from $\Phi(q_i)$ to a vertex of H (take γ_0 to be empty as $\Phi(q_0) = 1$).

The choice of M guarantees that there are distinct indices i, j such that the paths γ_i and γ_j have the same label $u \in F(X, Y)$. Writing c as the reduced word $c_1 \ldots c_t$ with $c_k \in X^{\pm 1}$, we have $c_1 \ldots c_i u$ and $u^{-1} c_{i+1} \ldots c_j u$ both represent elements of H, and

$$c =_G \underbrace{(c_1 \ldots c_i u)}_{\in H} \underbrace{(u^{-1} c_{i+1} \ldots c_j u)}_{\in H} \underbrace{(c_1 \ldots c_i u)}_{\in H}^{-1} (c_1 \ldots c_i c_{j+1} \ldots c_t)$$

$$=_G \underbrace{(c_1 \ldots c_i c_{i+1} \ldots c_j)(c_1 \ldots c_i)^{-1}}_{z \in H} (c_1 \ldots c_i c_{j+1} \ldots c_t) \ .$$

Thus $c =_A z(c_1 \ldots c_i c_{j+1} \ldots c_t)$, with $z \in A \times \{1\} \cap H$, $\ell(z) \leq 2M$. □

Remark

It is seems that by defining higher dimensional diagrams appropriately, the methods used here can be extended to show that a FP_n subgroup of a product of n free groups which is not $n-1$ dimensional is virtually a product of subgroups of the factors.

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