Graph small cancellation theory applied to prime alternating link groups

Rémi Cunéo and Hamish Short

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Abstract

We show that the Wirtinger presentation of a prime alternating link group satisfies a generalized small cancellation condition. This gives a simplification of Weinbaum's solution to the word and conjugacy problems for these groups, which extends to finite sums of alternating links.

1 Introduction

To show that the group of a prime alternating knot has solvable word and conjugacy problems, C.M.Weinbaum ([14], see also Lyndon and Schupp's book, [7, Chapter V]) showed that the Dehn presentation obtained from a elementary alternating projection of the knot verifies the small cancellation conditions $C(4) - T(4)$. Groups with such presentations have solvable word and conjugacy problems. Howerever the group presented is a free product of the knot group and an infinite cyclic group, which implies some technical detours in order for the solutions in the free product to give solutions in the knot group factor. Subsequently Appel and Schupp [2] extended the Dehn presentation method to all alternating knots, and Appel [1] used the usual Wirtinger presentation of the knot group and adapted the small cancellation methods to extend the result to some classes of non-alternating knots, "with considerable additional machinery".

A solution for the word problem for all knot groups was given by Waldhausen in 1968, and in 1993 Sela [11] solved the conjugacy problem for all knot groups. An alternative approach is given in [3, II.5.35], where it is shown that alternating link groups are groups of compact 2–dimensional piecewise–Euclidean 2–complexes of non–positive curvature (and mentioned that all link groups are fundamental groups of non–positively curved spaces).

The Wirtinger presentation, as Appel noted, does not verify any classical small cancellation conditions, but we shall show here that it does satisfy the conditions of a recent version of small cancellation theory, due to Gromov (see [5]) and Ollivier (see [9]) using graphs. The usual combinatorial arguments of classical small cancellation theory then give solutions to the word and conjugacy problems, and an additional observation extends the result to all finite sums of alternating links.

We begin by defining the presentation associated to a finite, labelled, oriented graph Γ, such that the generators are the labels on the edges, and the relators are the words read on the cycles of Γ. We describe Gromov–Ollivier's method of subdividing van Kampen diagrams into subdiagrams we call "megatiles" and consider small cancellation conditions on the graph presentation, which induce combinatorial restriction on the megatile diagrams. The usual small cancellation arguments then bound the number of megatiles in terms of the length of the boundary of the diagram, and an elementary lemma (see [9]) shows that each megatile satisfies an linear isoperimetric inequality. The solution of the word and conjugacy problems then follows.

In section 4, we apply the graph presentation method to prime alternating links. We use an elementary alternating projection (as in [14], as described in [7, Chapter V.8]), to obtain a dual graph whose group is the link group, and observe that this graph satisfies a $C_O(4) - T_O(4)$ condition (analagous to the $C(4) - T(4)$ condition used by Weinbaum).

Summarising, we obtain:

Theorem A. *Let L be a tame link in S^3, and let $P(L)$ be an elementary projection. Let Γ be the dual graph of $P(L)$, labelled by the Wirtinger (overcrossing) generators, and let $G(\Gamma)$ be the associated group.*

Then $G(\Gamma)$ is the group of the link, i.e. is isomorphic to $\pi_1(S^3 - L)$.
If $P(L)$ is an elementary alternating projection, then:
$G(\Gamma)$ satisfies the small cancellation conditions $C_O(4) - T_O(4)$.
The shortest relation in the generators has length 4.
The group has solvable word and conjugacy problems.

The groups of sums of such links can be obtained from disjoint unions of the graphs of the summands, in such a way that they also satisfy the $C_O(4)-T_O(4)$ condition, expressing the group of the sum as an amalgamated product along a meridian subgroup. Thus we obtain in section 5:

Theorem B. *Groups of finite sums of alternating links have graph presentations satisfying the conditions $C_O(4) - T_O(4)$, and so have solvable word and conjugacy problems.*

Note that such sums can be non–alternating. Note also that all the generators of the group $G(\Gamma)$ are meridians, and so if the alternating projection contains at least one crossing, the shortest relation has length 4, and the group is not cyclic (see also [8]) (else all the meridians would be equal, implying relations of length 2). With a little more work, one can show that prime alternating link presentations with at least one crossing correspond to non-trivial links, as they contain commuting elements without common powers.

We end with some observations on the complexity of the solutions, OR NOT, AS THE CASE MAY BE.

Most of the results presented here appear in the thesis [4] of the first author. We would like to thank François Dahmani and Paul Schupp for their comments on the thesis, and the referee for his/her helpful remarks, which have significantly improved our presentation.

2 Graph presentations, diagrams, megatiles

We summarize here some basic definitions and properties of presentations, van Kampen diagrams and small cancellation theory from this new viewpoint: for more details, see [9] and [4].

Throughout, Γ will be a finite, oriented, not necessarily connected graph. The (oriented) edges of Γ are labelled by letters, which form a set S. The same label will usually occur on several different edges. We shall suppose that Γ is reduced: at each vertex v of Γ, and for each label $a \in S$, there is at most one oriented edge beginning at v labelled a, and at most one oriented edge labelled a terminating at v. This means that there is no path in Γ labelled aa^{-1} or $a^{-1}a$ with $a \in S$ (other than a round trip up and down a single edge). We also suppose that Γ has no redundant cycles, i.e. Γ does not contain two distinct cycles with the same label.

The group $G(\Gamma)$ associated to Γ is the free group $F(S)$ on the set of labels, quotiented by the normal subgroup N normally generated by the words on all the cycles of Γ. Any choice $\mathcal{R}_\Gamma$ of labels on a finite set of cycles in Γ generating $H_1(\Gamma)$ gives a finite presentation $\langle S \mid \mathcal{R}_\Gamma \rangle$ of $G(\Gamma)$. For instance for $\mathcal{R}_\Gamma$ we can take labels on a basis (or generating family) of cycles (immersed circles) or circuits (embedded circles) of Γ. An example is given in figure 2 below. Another possible choice for $\mathcal{R}_\Gamma$ is the set of all labels on cycles of length at most $k.\mathrm{diam}(\Gamma)$, for any $k \geq 2$.

Classical small cancellation theory for a finite presentation $\langle S \mid \mathcal{R} \rangle$ uses conditions on the length of a piece relative to the lengths of relators containing it, where a piece is a common initial subword of two distinct elements of $\mathcal{R}^C$, the set of all cyclic conjugates of elements of $\mathcal{R} \cup \mathcal{R}^{-1}$. In the "graph small cancellation theory" used here, a piece relative to the labelled graph Γ is defined as follows:

Definition 1. *A* **piece** *is a word labelling two distinct paths immersed in* Γ.

For a finite presentation $\langle S \mid r_1, \ldots, r_n \rangle$, form a graph consisting of n disjoint loops L_i, subdivided into $|r_i|$ edges, oriented and labelled so that the word read from a suitable base point is the word r_i. As usual, $|w|$ denotes the length of the word w (in the free group or semigroup $F(S)$, according to the context). With this graph, the two concepts of piece coïncide (if no r_i is a proper power).

A simple example illustrates the difference between the two approaches. In classical small cancellation theory, in the presentation $\mathcal{P} = \langle a, b, c \mid ba^{-1}, bc^{-1} \rangle$, b is a piece of length 1, and has half the length of the relators containing it. Let Γ be a θ−curve — two vertices joined by three edges, labelled by a, b and c. As each of a, b and c has an unique immersion in Γ, there is no piece in Γ. It is clear that $\mathcal{P}$ is a presentation for $G(\Gamma)$.

A (van Kampen) diagram over a finite presentation $\mathcal{P} = \langle S \mid \mathcal{R} \rangle$ is a finite, planar, connected, simply connected 2-complex D, with oriented edges labelled in S, such that the boundary of each bounded face is labelled by a word of $\mathcal{R}$ (up to cyclic permutation and inversion). The label on the outer boundary (the boundary of the complement of D in $\mathbb{R}^2$) is a word w in the free semigroup on $S \cup S^{-1}$. We also say that D is a diagram for w over $\mathcal{P}$ (see for instance [7, chapter V] or [12] for more about diagrams). A diagram for w is minimal if all other diagrams for w have at least as many faces. Van Kampen proved (see [13]) that every word w in $\langle\langle \mathcal{R} \rangle\rangle$ has a diagram over $\mathcal{P}$. We define the area of $w \in \langle\langle \mathcal{R} \rangle\rangle$, $Area_{\mathcal{P}}(w)$, to be the number of 2–cells in a minimal diagram for w over $\mathcal{P}$. We say that the presentation satisfies a linear (quadratic, cubic, exponential, recursive) isoperimetric inequality if there is a linear (quadratic, cubic, exponential, recursive) function $f_{\mathcal{P}} : \mathbb{N} \to \mathbb{R}$ such that $Area(w) \leq f_{\mathcal{P}}(|w|)$. A standard result of this domain is that the word problem of a presentation is solvable if and only if it satisfies a recursive isoperimetric inequality (see for instance [12, Thm 1.1]).

For the graph Γ, we fix a finite set of labels $\mathcal{R}_\Gamma$ on cycles generating $H_1(\Gamma)$. Let D be a minimal diagram over the finite presentation $\langle S \mid \mathcal{R}_\Gamma \rangle$

of the group $G(\Gamma)$. Each face F of D is labelled by a word $r \in \mathcal{R}_\Gamma$. The word r is the label on a unique cycle c in Γ (uniqueness follows from the non–redundancy of Γ). There is thus a simplicial immersion from ∂F to Γ: we say that ∂F lifts to Γ.

Definition 2. *Let F_1, F_2 be two faces of D, and let e be an edge in the intersection of their boundaries in D. Let $\phi_i : \partial F_i \to \Gamma, i = 1, 2$ be lifts of the boundaries of the two faces. We say that e is an **edge originating from** Γ if the images of the two lifts $\phi_1(e), \phi_2(e)$ of e to Γ are the same edge of Γ. The two faces F_1 and F_2 are said to be* **Γ-adjacent***.*

*The reflexive and transitive closure of the Γ-adjacency relation is an equivalence relation. Each equivalence class gives rise to a **megatile** M as follows: there is a closed 2–cell for each face in the equivalence class, and the edges originating from Γ common to Γ-adjacent faces are identified.*

Seen as an abstract complex, each megatile M has a boundary ∂M, which is a not necessarily connected 1-complex. The rule for identifiying edges of faces in M means that the 1-skeleton of M lifts to Γ, and in particular each component of ∂M lifts to Γ. There is obvious map from M to D. Notice that edges of ∂M do not originate from Γ, and distinct edges in ∂M may be identified in D (see figure 1). That is, M is not necessarily homeomorphic to the closure of the open faces of D which make up M. In figure 1, the lower-left megatile M of D is a disc, with boundary ∂M a circle, but the closure of M in D is an annulus.

This notion of megatile is implicitly used in [9] (where faces are called "tiles"). Notice that each face of a minimal diagram belongs to one and only one megatile, and that a megatile of a minimal diagram D is not necessarily simply connected (see the figure 1). We shall give conditions below (theorem 9), that will ensure that each megatile of a minimal diagram is simply connected (two megatiles of figure 1 are impossible under these conditions).

The megatile diagram of D, denoted by D', is the 2-complex, unique by construction, built from D, by deleting open edges of D originating from Γ and vertices of D meeting only such edges (see figure 1). Each megatile naturally maps to a face of D', which, abusing language, we shall continue to refer to as a megatile. As is common in these diagrams, we supress vertices of degree 2 (see [7] p 242). In a diagram D for a cyclically reduced word, we can distinguish a D-edge, which is an edge of D labelled by a generator in S (or its inverse), and a D'-edge, which is an arc of the megatile diagram D' of

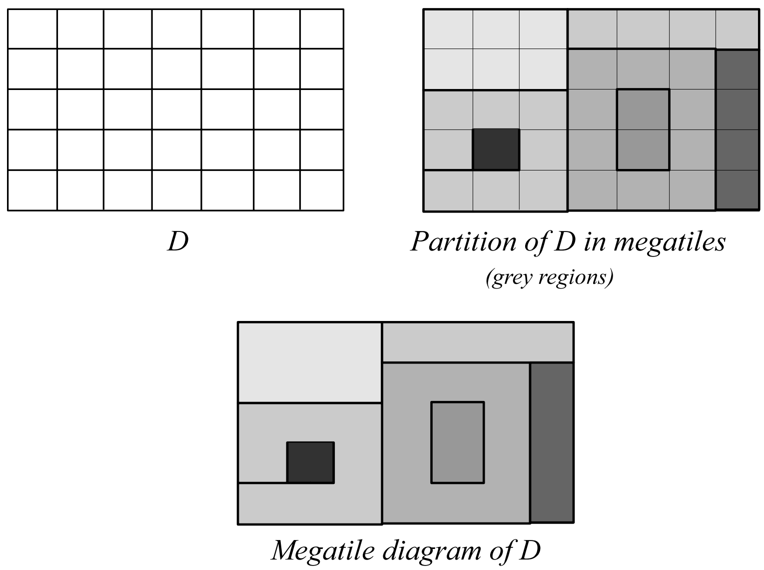

Figure 1: Megatile diagram of a diagram D

D, labelled by a word (as vertices of degree two have been supressed). The endpoints of a D'-edge have degree at least 3 in D'.

We repeat that the important property of megatiles is that they are intuitively maximal lifts of the one-skeleton of D to Γ, and for each megatile M, every component of ∂M lifts to a cycle in Γ.

3 Results in graph small cancellation theory

As usual, Γ is a finite, labelled, oriented graph (not necessarily connected) satisfying the conditions given earlier. We give two important lemmas about megatiles. The first one comes from [9]:

Lemma 3. *Let $\mathcal{R}_\Gamma$ be a finite set of labels for some generating set for $H_1(\Gamma)$. There is a constant $C > 0$ such that for any label w on a cycle in Γ, there is a diagram for $w = 1$ over $\langle S \mid \mathcal{R}_\Gamma \rangle$ of area bounded by $C|w|$.*

Proof. Let $\mu = \max\{Area(w) \mid |w| \leq 3\Delta,\ w \text{ labels a cycle in } \Gamma\}$ where $\Delta = \max\{d(x,y) \mid x, y \text{ in the same component of } \Gamma\}$ is the diameter of Γ.

If $|w| > 3\Delta$ then $w = w_1 w_2$, where $|w_2| = 2\Delta$. The endpoints of w_2 in Γ are in the same component, at distance at most Δ, so there is a path in Γ of length at most Δ joining them, with label w' say, such that $w_2 w' = 1$ in $G(\Gamma)$ and there is a diagram D'' for $w_2(w')^{-1} = 1$ of area at most μ. By induction there is a diagram D' of area at most $C|w_1 w'|$ for $w_1(w')^{-1} = 1$, and so there is a diagram D for $w_1 w_2 = 1$ obtained by identifying the segments labelled w' of the boundaries of D' and D''. Thus $Area(D) = Area(D') + Area(D'')$ and

$$Area(D) \leq C(|w_1| + |w'|) + \mu \leq C|w_1| + C(|w_2| - \Delta) + \mu$$

which is less than $C|w|$ if $C\Delta > \mu \iff C > \frac{\mu}{\Delta}$. □

If M and M' are two megatiles of the megatile diagram D' coming from a diagram D over a finite presentation for $G(\Gamma)$, each arc of $\partial M \cap \partial M'$ is a piece (not necessarily maximal, i.e. included in a longer piece of Γ). As the boundaries of megatiles lift to cycles in Γ, we have:

Corollary 4. *Simply connected megatiles satisfy a linear isoperimetric inequality.*

Here the length of the boundary of a megatile can be measured either in $F(S)$ or in terms of pieces, at the expense of changing the constant by a factor (the maximal length of a piece).

Definition 5. *We say that the graph Γ verifies the $C_O(p)$ condition when no cycle of Γ can be decomposed into fewer than p pieces (with disjoint interior).*

Let $q \geq 3$. We say that Γ verifies the $T_O(q)$ condition when: if there are h ($3 \leq h < q$) paths (of length 2) in Γ labelled $p_1 p_2^{-1}, p_2 p_3^{-1}, ..., p_h p_1^{-1}$, with $p_1, p_2, ..., p_h$ generators (or their inverses, and $p_i \neq p_{i+1}$ (mod h) as Γ reduced), which lift to Γ, then the h paths have the same vertex of Γ as their midpoint.

These last technical definitions imply more simple and convenient properties. A megatile (resp. vertex, or edge) of a diagram D is *internal* if its intersection with the boundary ∂D contains no edge (resp. is not in ∂D).

Properties 6. *Let D be a minimal diagram over Γ for some choice $\mathcal{R}$ of relators, and let D' be the megatile diagram of D. Then:*

If Γ satisfies $C_O(p)$, the boundary of each internal simply connected megatile M of D' is composed of at least p D'-edges.

If Γ satisfies $T_O(q)$, the degree of each internal vertex of D' is at least q.

Following [7], a connected planar complex such that all faces are simply connected, is a (p, q) map if every internal face has degree at least p, and every internal vertex has degree at least q. We are concerned with $(4, 4)$ maps here, and the essential property used is that such maps contain faces with 2 or 3 edges, one of which is necessarily a boundary edge.

If γ is a path in the 1-skeleton of a 2-complex, then we denote by $\ell(\gamma)$ the number of edges traversed by γ (counted with multiplicity).

Lemma 7. *Let Q be a $(4, 4)$-map without vertices of degree 2 which is a topological disk.*

The number of faces is at most $4\ell(\partial Q)^2$, and the number of edges in Q is at most 3 times the number of faces.

Proof. The results hold if Q has just two faces, so suppose that Q has at least 3 faces.

The first inequality is well–known, and follows for instance from the Area Theorem [7, V.6.2] (and is stated in the proof of V.6.3).

To prove the second, note that Q contains a face F with 2 or 3 sides, i.e. a face with just one edge f on the boundary, and 1 or 2 internal edges. Removing the boundary edge f, and suppressing if necessary any vertex of degree two which arises (there can be at most two), gives a map Q', with one less face, and either one, two or three fewer edges.

If the face F has 2 edges, then Q' is also a topological disk, and the result follows by induction on the number of faces.

If the face F has 3 edges, then it is possible for Q' to be two topological discs joined at a vertex. Using induction as before and summing over the two disk components, the result follows.

□

This result generalises to maps which are not topological disks:

Lemma 8. *Let Q be a $(4, 4)$-map (with no vertices of degree 2).*

The number of faces is at most $4\ell(\partial Q)^2$, and the number of edges in Q is at most $13\ell(\partial Q)^2$.

Proof. If Q is not a topological disc, it consists of a collection of topological disks joined in a tree-like manner. The edges joining the disk components of Q are counted in $\ell(\partial Q)$, but do not contribute to the area, so the first part follows.

For the second part, the previous result gives the number of edges in each disk component, and then adding $\ell(\partial Q)^2 \geq \ell(\partial Q)$ takes into account the edges joining the disk components. $\square$

Theorem 9. *Suppose* Γ *satisfies* $C_O(4) - T_O(4)$*.*

1. *If* D *is a minimal diagram over* Γ *(for some choice of relators* $\mathcal{R}_\Gamma$*), and* D' *is the megatile diagram of* D*, then each megatile of* D *is simply connected, and* D' *is a* $(4,4)$ *map.*
2. *The group* $G(\Gamma)$ *satisfies a quadratic isoperimetric inequality.*
3. *The word and conjugacy problems are solvable for* $G(\Gamma)$*.*

Analogous results can be proved for the $C_O(6)$ and $C_O(3) - T_O(6)$ cases.

Proof. 1. Recall that D is a simply connected planar complex, and so is the megatile diagram D'. If the megatiles of D' are simply connected, then as noted above the properties $C_O(4) - T_O(4)$ ensure that D' is a $(4,4)$ map.

Suppose that there are non–simply connected megatiles in D'. In the complement of the non–simply connected megatiles of D', let D'' be an innermost topological disk. Let M be the non-simply connected megatile such that $\partial D''$ is a connected component of ∂M. The megatiles in D'' are all simply connected, by the innermost property, so that the megatile diagram contained in D'' is a $(4,4)$ map. The essential property of $(4,4)$ maps is that there is a face with at most 3 edges, necessarily having one edge on the boundary. This means that there is a megatile M' in D'' meeting $\partial M \subset \partial D''$ in an arc which is strictly longer than a piece, so that M' and M contain in their common boundary a segment with a unique lift to Γ. Thus the lift of M to Γ extends to M', and M and M' form part of the same megatile, giving a contradiction.

2. Choose a finite set of relators $\mathcal{R}_\Gamma$ giving a presentation $\mathcal{P} = \langle S \mid \mathcal{R}_\Gamma \rangle$ for the group $G(\Gamma)$. We show that $\mathcal{P}$ satisfies a quadratic isoperimetric inequality; changing to another finite presentation merely changes the constants involved. Let w be a cyclically reduced word in the generators which is a relation, and let D be a minimal diagram for $w = 1$ over $\mathcal{P}$. By part 1, the megatile diagram D' obtained from D is a $(4,4)$ map without vertices of degree 1 or 2. By Lemma 8, the number of faces in D' is at most $4\ell(\partial D')^2 \leq 4|w|^2$, and the number of D'-edges is at most $13|w|^2$. Each face of D' corresponds to a megatile M, and each diagram on a megatile M,

$D \cap M$, satisfies a linear isoperimetric inequality, $Area(D \cap M) \leq C.\ell_S(\partial M)$ (by Lemma 3). Here $\ell_S(\partial M)$ denotes the length of the word labelling the boundary of the subdiagram $D \cap M$ in the megatile M, and $\ell(\partial M)$ is the number of D'-edges in ∂M.

Summing over the diagrams in each megatile:

$$Area(D) = \sum_{M \text{megatile of } D'} Area(D \cap M) \leq \sum_M C.\ell_S(\partial M) = C \sum_M \ell_S(\partial M)$$

This sum is at most twice the number of D'–edges (internal edges are counted twice) in the megatile diagram, but measured in the original D–lengths. Each edge of D' is either an internal edge, and therefore a piece, or is contained in the boundary of D'. There are a finite number of pieces in Γ, so there is a maximal length of a piece, say ρ. The length of the boundary of D' is bounded above by $|w|$, the length of the boundary of D.

Seperating the edges of ∂M into those that are internal in D', and those that lie on $\partial D'$, we see that:

$$\sum_M \ell_S(\partial M) \leq 2 \sum_{e \text{ internal } D'-\text{edge}} \ell_S(e) + \sum_{f \text{ edge in } \partial D'} \ell_S(f)$$

The number of D'–edges is at most $13\ell(\partial D')^2 \leq 13|w|^2$. Thus

$$\sum_M \ell_S(\partial M) \leq 26\rho|w|^2 + |w| \leq 27\rho|w|^2$$

and the quadratic isometric inequality is established.

3. Having a recursive isoperimetric inequality is equivalent to having a solvable word problem. The fact that the megatile diagrams are $(4,4)$–maps can then be used to solve the conjugacy problem, as in [7, V.7.4]. □

4 Application to prime alternating link groups

a. Construction of a link graph

One considers an oriented link L embedded in $\mathbb{S}^3$ and a regular projection $P(L)$ of this link. As is usual, each double point of this projection, two opposite germs of the arcs are identified to signify the presence of an overcrossing and an undercrossing.

One orients and labels each overcrossing arc (of each component) between two crossings of this projection (in the same sense) by a generator; these generators correspond to those of the Wirtinger presentation. To obtain the Wirtinger presentation of the link group, a relator of length 4 is found at each crossing (see for instance [10] pp 56-59). The graph $\Gamma(L)$ (or simply Γ) of the link is obtained by duality from the regular projection. To each open and connected region R of the regular projection, we associate a vertex $v(R)$ of the graph and to each arc of the regular projection separating two regions R_1, R_2 we associate an edge of the graph joining the vertices $v(R_1), v(R_2)$ transverse to the arc. One then agrees on a rule for the labelling and the orienting of the graph:

If one imagines the link projected onto the graph, seen from above, and if one follows each component of the link in the positive sense, one labels each edge of the graph by the same generator which labels the transverse overcrossing arc of the link, oriented, by convention, from left to right.

At each double point of the regular projection (a double point corresponds to an overcrossing of the link projection), there is in general a circuit of length 4 in Γ, where one generator conjugates another to a third. This is a Wirtinger relator (see [10] p 56-59).

An example of this construction is given in figure 2.

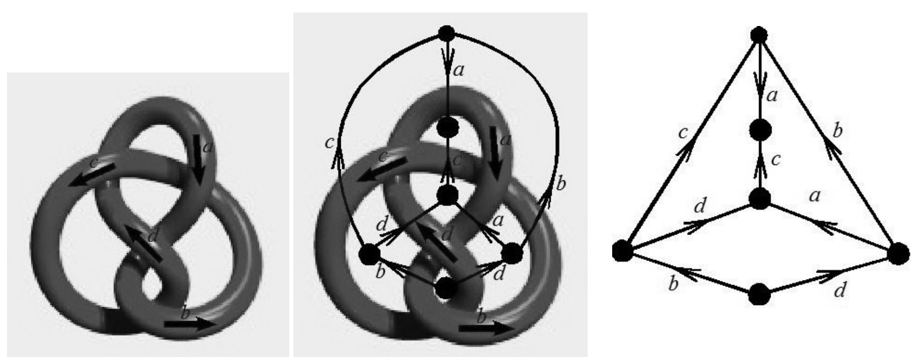

Figure 2: Graph associated to (the regular projection of) the figure eight knot giving $\mathcal{P}(\Gamma) = \langle a, b, c, d \mid bcb^{-1}d^{-1}, cac^{-1}d^{-1}, dad^{-1}b^{-1} \rangle$, when $\mathcal{R}_\Gamma$ is chosen to be the set of labels on circuits of length 4, one for each of three of the four overcrossings. See [10] page 58 where $x_1 = b$, $x_2 = a$, $x_3 = d$, $x_4 = c$.

If there is a component of the link projection which is a circle with no crossings, then the link splits, and the group has a free cyclic factor. This means that some labels do not appear in relators obtained from cycles in the graph Γ. As the group of a split link is a free product of non–split link groups, and the solutions of the word and conjugcay problems in the factors give solutions in the free product, we can assume that the links concerned are non–split and non–trivial.

b. Word and conjugacy problems

Following [7, V.8.2] it is easy to see that a prime alternating link has an alternating projection that is elementary: each double point is on the boundary of four distinct regions, and that no two regions have two edges in the intersection of their boundaries. For the latter, note that if there were two such edges, then there is an embedded $\mathbb{S}^1$ in the plane, meeting the link projection in two points; and the link is not prime (if the other condition holds); see figure 3.

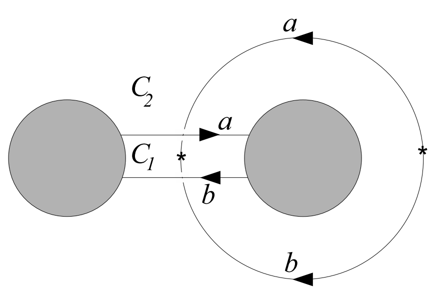

Figure 3: Two regions meeting in two arcs means non–prime

Theorem 10. *Let Γ be the graph associated to an elementary alternating link projection $P(L)$ as above. Then:*

Γ *satisfies the small cancellation conditions $C_O(4) - T_O(4)$.*

A suitable choice of $\mathcal{R}_\Gamma$ gives the Wirtinger presentation of the link group.

Prime alternating links have elementary alternating projections, so:

Corollary 11. *Groups of prime alternating links have solvable word and conjugacy problems.*

Proof In the graph obtained from an alternating projection, each generator occurs exactly twice, on the two edges corresponding to an overcrossing.

More precisely, as in figure 4, the two edges labelled a in the graph Γ, occur in a cycle of length 4, are separated by two other edges and have opposite orientation (in this cycle).

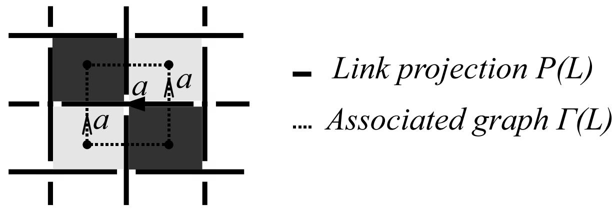

Figure 4: Each generator occurs exactly twice in the link graph (and once in each of two other circuits of length 4), in the same cycle of length 4, "alternately" and with opposite orientation.

We see easily that the only possible pieces are of length 1. Indeed, the only way to have a piece of length 2 in the graph is shown in figure 5 and this implies that the link is not alternating.

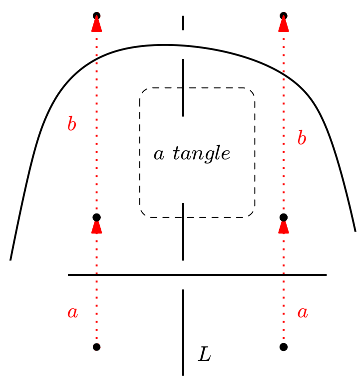

Figure 5: Piece of length 2 in the graph

For example, in the graph of the figure eight knot (figure 2), one can read the words $a^{-1}b$ (or $b^{-1}a$) and ba (or $a^{-1}b^{-1}$) in the graph in just one place, and no word $a^{\pm1}b^{\pm1}$ appears elsewhere in the graph. In this example it is clear that all the cycles of Γ are of length at least 4.

In general, the regular projection of a link L gives a checkerboard decomposition of the plane, so that the dual graph Γ is a bipartite graph. Therefore the cycles of Γ have even length (each cycle passes alternately from a black region to a white region and so on). When the projection is elementary, there are no cycles of length 2 in Γ, and when the projection is alternating, there are no pieces of length 2. Thus no cycle in Γ can be decomposed into less than 4 pieces, and the $C_O(4)$ condition holds.

$T_O(4)$ condition:

With the convention of erasing vertices of degree 2 in diagrams (see [7] p 242), we must verify that in a minimal diagram D over Γ, every internal vertex of degree 3 is internal to a megatile of D.

In the example of the figure eight knot, we just list all possible configurations of vertices of degree 3 to realise that this is the case.

For the general case, let v be a vertex of degree 3, internal to D'', the subdiagram of D composed of three faces F_{ab}, F_{bc} and F_{ac} of Γ ($\{v\} = \partial F_{ab} \cap \partial F_{bc} \cap \partial F_{ac}$). We use the notations of figure 6.

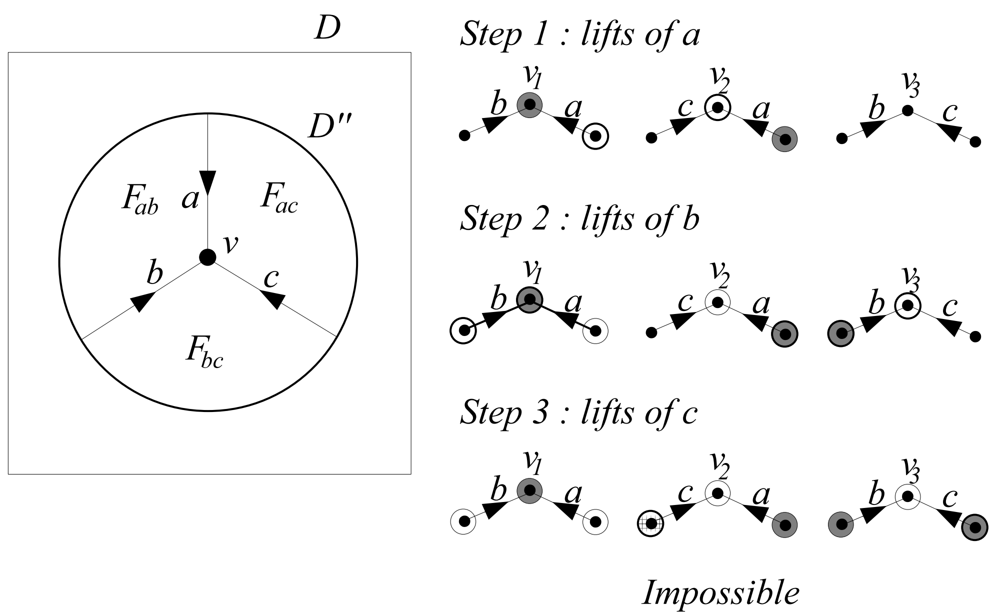

Figure 6: Vertex of degree 3 in a minimal diagram

Case 1: at least one of the three edges a, b, c at v originates from Γ. Without loss of generality, suppose the edge is labelled b, then (F_{ab}, F_{bc}) is (or is included in) a megatile of D and the tripod $(a; b; c)$ of vertex v lifts entirely to Γ (with the vertex v). As ac^{-1}, of length 2, is not a piece, the boundary of the face F_{ac} lifts together with the unique lift of the path labelled by this word in Γ. It follows in this case that D'' is contained in a megatile of D, the three edges originate from Γ and v is an internal vertex of a megatile of D.

Case 2: none of the edges a, b, c originates from Γ.
Each of the three letters a, b, c has exactly two distinct lifts in Γ.

As the pieces of Γ are of length 1, there cannot be two distinct lifts of the paths labelled by ab^{-1}, bc^{-1}, ca^{-1} (or their inverses), so v lifts to Γ as an internal vertex of each of these three paths.

But a (for example) appears exactly twice in Γ in a circuit of length 4, with opposite orientations, separated by edges not labelled by a (see figure 4).

We use a colouring argument (see figure 6):

One colours vertices of Γ alternately in black and white (from the checkerboard colouring of the projection). According to the previous remark, the two initial points (similarly, the two terminal points) of the edge labelled a have different colours in Γ. Likewise for b and c.

The vertex v (of the figure 6) has three distinct lifts v_1, v_2 and v_3 in Γ. As the edge of D labelled by a is on the boundary of two megatiles of D, it lifts necessarily to two different places in Γ. By doing the same with b, then with c, we get a contradiction on colourations: we have the same colouring for both initial points of the two distinct edges of the graph labelled by c. This is impossible. So every internal vertex of degree 3 is included in a megatile.

5 Sums of prime alternating links

The group of the sum of two links is the free product of the two link groups, amalgamated along a cyclic subgroup generated by a meridian. We show that the group has a graph presentation obtained from the disjoint union of graphs for each component, with a slight change in the relabelling. We show then that this graph satisfies a $C_O(4) - T_O(4)$ condition when the two component graphs do.

Let Γ_1, Γ_2 be labelled graphs, with disjoint sets of labels X_1, X_2, and associated groups G_1, G_2. Suppose that every label in $X_1 \cup X_2$ is a piece, and choose $a_1 \in X_1$ and $a_2 \in X_2$ such that no path in Γ_i is labelled by a_i^2, $i = 1, 2$. Let Γ_2' be the graph obtained from Γ_2 by replacing all occurrences of the label a_2 by the label a_1.

It is easy to see that the group associated to the disjoint union $\Gamma_3 = \Gamma_1 \sqcup \Gamma_2'$ is the amalgamated product $G_1 *_{\mathbb{Z}} G_2$ identifying the generators a and a'.

Lemma 12.

1. $G(\Gamma_3) \cong G_1 *_{a_1=a_2} G_2$.
2. *If* Γ_1 *and* Γ_2 *satisfy* $C_O(6)$, *(resp.* $C_O(4)$*) then so does* Γ_3*;*
3. *If* Γ_1 *and* Γ_2 *satisfy* $T_O(4)$ *then so does* Γ_3.

Proof. (1) Let C_i be a finite set of cycles generating $H_1(\Gamma_i)$, for $i = 1, 2$. and let $\mathcal{R}_i$ be the labels on the cycles of C_i. The groups G_i then have presentations $\langle X_i \mid \mathcal{R}_i \rangle$ for $i = 1, 2$. As Γ_3 is the disjoint unioin of Γ_1 and Γ_2, the set $\mathcal{R}_1 \cup \mathcal{R}_2$ is a finite set of cycles generating $H_1(\Gamma_1 \cup \Gamma_2)$, and $\langle X_1, X_2 \mid \mathcal{R}_1, \mathcal{R}_2 \rangle$ is a finite presentation for the group $G(\Gamma_1 \cup \Gamma_2) \cong G_1 * G_2$. Replacing the label a_2 by the label a_1 corresponds to Tietze transformations on the presentation $\langle X_1, X_2 \mid \mathcal{R}_1, \mathcal{R}_2, a_1 = a_2 \rangle$ of the amalgamated product $\langle X_1 \mid \mathcal{R}_1 \rangle *_{a_1=a_2} \langle X_2 \mid \mathcal{R}_2 \rangle$, to give a graph presentation of $G(\Gamma_3)$.
(2) The words in $X_1 \cup (X_2 - \{a_2\})$ which can label paths in both Γ_1 and Γ_2' are powers of a_1, the only label occuring in both components. The condition on the choice of a_1, a_2 means that no path in Γ_3 is labelled by a_1^2. Thus the only new pieces in Γ_3 are those from Γ_2 obtained by replacing the a_2 label by a_1. Thus all pieces satisfy the same $C_O(6)$ (resp. $C_O(4)$) condition as before.
(3) The only way for the $T_O(4)$ condition to fail in Γ_3, while at the same time holding in Γ_1 and in Γ_2, would be for the sequence of paths $p_1 p_2^{-1}, p_2 p_3^{-1}, p_3 p_1^{-1}$ in the definition of the $T_O(4)$ condition to contain paths in both of the components of Γ_3, else the condition would fail in the component concerned. But no path of length two lifts to both components, and no path is labelled a_1^2, so such a non-trivial sequence cannot have length 3. □

It is possible to generalise the above result to amalgamate over subgroups generated by larger sets of elements, but we restrict to this case as it has the following immediate application:

Corollary 13. *The group of a finite sum of alternating links has a presentation satisfying the* $C_O(4) - T_O(4)$ *condition, and so has solvable word and conjugacy problems.*

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Remi.Cuneo@free.fr
Hamish.Short@latp.univ-mrs.fr

L.A.T.P. UMR 7353, Université de Aix–Marseille,
37 Rue Joliot Curie, 13453 Marseille, France