Objective: To present discrete functional analysis tools for proving the convergence of numerical schemes, mainly for parabolic equations (Stefan problem, incompressible and compressible Navier-Stokes equations)

Works with many co-authors
First example, compressible Navier-Stokes Equations

\( \Omega \) : bounded open connected set of \( \mathbb{R}^3 \)

\( T > 0, \gamma > 3/2, f \in L^2([0, T[, L^2(\Omega)) \)

\[
\begin{align*}
\partial_t \rho + \text{div}(\rho u) &= 0, \\
\partial_t (\rho u) + \text{div}(\rho u \otimes u) - \Delta u + \nabla p &= f, \\
p &= \rho^\gamma.
\end{align*}
\]

Dirichlet boundary condition : \( u = 0 \)

Initial condition on \( \rho \) and \( u \) (or on \( \rho u \)).
First example, compressible Navier-Stokes Equations

\[ \Omega: \text{bounded open connected set of } \mathbb{R}^3 \]
\[ T > 0, \quad \gamma > 3/2, \quad f \in L^2(]0, T[, L^2(\Omega)) \]

\[ \partial_{n,t} \rho + \text{div}_n(\rho_n u_n) = 0, \]
\[ \partial_{n,t}(\rho_n u_n) + \text{div}_n(\rho_n u_n \otimes u_n) - \Delta_n u_n + \nabla_n p_n = f_n, \]
\[ p_n = \rho_n^\gamma. \]

- Estimates on \( u_n, \rho_n, p_n \)
- Passing to the limit on \( \rho_n u_n \) and \( \rho_n u_n \otimes u_n \)
- Passing to the limit on \( p_n = \rho_n^\gamma \).

For nonlinear terms, weak convergences are not sufficient
Second example, Stefan problem

\( \Omega \): bounded open connected set of \( \mathbb{R}^3 \), \( T > 0 \)

\( \partial_t \rho - \Delta u = 0, \ u = \varphi(\rho) \)

\( \varphi \in C(\mathbb{R}, \mathbb{R}) \) is nondecreasing \( \varphi' = 0 \) on \( ]a, b[ \), \( a < b \)

Dirichlet boundary condition: \( u = 0 \)

Initial condition on \( \rho \)
Second example, Stefan problem

$\Omega$ : bounded open connected set of $\mathbb{R}^3$, $T > 0$

$\partial_{n,t} \rho_n - \Delta_n u_n = 0$, $u_n = \varphi(\rho_n)$

$\varphi \in C(\mathbb{R}, \mathbb{R})$ is nondecreasing $\varphi' = 0$ on $]a, b[$, $a < b$

- Estimates on $u_n$, $\rho_n$
- Passing to the limit on the equation, $\partial_t \rho - \Delta u = 0$
- Prove $u = \varphi(\rho)$

First step : Prove that $\int_{]0,T[\times\Omega} \rho_n u_n \to \int_{]0,T[\times\Omega} \rho u$

Second step : Minty trick, $u = \varphi(\rho)$
Common difficulty for this two examples

$\Omega$: bounded open connected set of $\mathbb{R}^3$, $T > 0$,

$\rho_n \to \rho$ weakly in $L^2(0, T, L^q(\Omega))$

$u_n \to u$ weakly in $L^2(0, T, L^p(\Omega))$

$1 < p, q < +\infty$, $\frac{1}{p} + \frac{1}{q} = 1$

Question: $\int_0^T \int_\Omega \rho_n u_n \to \int_0^T \int_\Omega \rho u$ ?

In general, no. We need an additional hypothesis
Continuous setting, Stationary case

Discrete setting mimics continuous setting.

$\Omega$ bounded open set of $\mathbb{R}^3$. $1 < p, q < +\infty$, $\frac{1}{p} + \frac{1}{q} = 1$

$\rho_n \rightarrow \rho$ weakly in $L^q(\Omega)$
$u_n \rightarrow u$ weakly in $L^p(\Omega)$

Question: $\int_\Omega \rho_n u_n \rightarrow \int_\Omega \rho u$ ?

- in general, no.
- yes if $(u_n)_n$ is bounded in $H^1_0(\Omega)$ and $p < 6$

Two methods,

- Compactness on $(u_n)_n$ (M1)
- Compactness on $(\rho_n)_n$ (M2)
Continuous setting, Stationary case, M1

$\Omega$ bounded open set of $\mathbb{R}^3$, $1 < p < 6$, $q = p/(p - 1)$

$\rho_n \to \rho$ weakly in $L^q(\Omega)$

$u_n \to u$ weakly in $L^p(\Omega)$

$(u_n)_n$ is bounded in $H^1_0(\Omega)$

Compact embedding of $H^1_0(\Omega)$ in $L^p(\Omega)$

Then

$u_n \to u$ in $L^p(\Omega)$

$\rho_n \to \rho$ weakly in $L^q(\Omega)$

and $\int_{\Omega} \rho_n u_n \to \int_{\Omega} \rho u$
Continuous setting, Stationary case, M2

\( \Omega \) bounded open set of \( \mathbb{R}^3 \), \( 1 < p < 6, \ q = p/(p-1) \)

\( \rho_n \to \rho \) weakly in \( L^q(\Omega) \)
\( u_n \to u \) weakly in \( L^p(\Omega) \)
\( (u_n)_n \) is bounded in \( H^1_0(\Omega) \)

Identify \( L^2(\Omega)' \) with \( L^2(\Omega) \)

Compact embedding of \( L^q(\Omega) \) in \( H^{-1}(\Omega) \)

Then
\( u_n \to u \) weakly in \( H^1_0(\Omega) \)
\( \rho_n \to \rho \) in \( H^{-1}(\Omega) \)

and \( \int_{\Omega} \rho_n u_n = \langle \rho_n, u_n \rangle_{H^{-1},H^1_0} \to \langle \rho, u \rangle_{H^{-1},H^1_0} = \int_{\Omega} \rho u \)
Discrete setting, stationary case

It is possible to adapt the previous methods to a discrete setting where $H^1_0(\Omega)$ is replaced by a space $H_n$ which depends on $n$ (with a norm, depending on $n$, “close” to the $H^1_0$-norm).
Space discretization, Finite Volume scheme

Mesh $\mathcal{M}$.

$\sigma = K|L$

$\mathcal{M}$: functions from $\Omega$ to $\mathbb{R}$, constant on each $K$, $K \in \mathcal{M}$

Figure: Here is an example of *admissible mesh*

$\text{size}(\mathcal{M}) = \sup\{\text{diam}(K), K \in \mathcal{M}\}$
Discrete $H^1_0$-norm

Mesh: $\mathcal{M}$ (not necessarily admissible)

$u \in H_\mathcal{M}$ (that is $u$ is a function constant on each $K$, $K \in \mathcal{M}$).

$$\|u\|_{1,2,n}^2 = \sum_{\sigma \in \mathcal{E}_{\text{int}}, \sigma = K|L} m_\sigma d_\sigma \left| \frac{u_K - u_L}{d_\sigma} \right|^2 + \sum_{\sigma \in \mathcal{E}_{\text{ext}}, \sigma \in \mathcal{E}_K} m_\sigma d_\sigma \left| \frac{u_K}{d_\sigma} \right|^2.$$
Discrete setting, Stationary case, M1

\( \rho_n, u_n \in H_{M_n}, \text{ size}(M_n) \to 0 \text{ as } n \to \infty \) (regularity of the meshes)

\( p = q = 2 \) (for simplicity)

\( \rho_n \to \rho \) weakly in \( L^2(\Omega) \)

\( u_n \to u \) weakly in \( L^2(\Omega) \)

\((u_n)_n\) is bounded in \( H_{M_n}, \| \cdot \|_{1,2, M_n} \)

“Compact embedding” of \((H_{M_n}, \| \cdot \|_{1,2, M_n})_n\) in \( L^2(\Omega) \)

Then

\( u_n \to u \) in \( L^2(\Omega) \)

\( \rho_n \to \rho \) weakly in \( L^2(\Omega) \)

and \( \int_{\Omega} \rho_n u_n \to \int_{\Omega} \rho u \)

Admissible meshes: Compactness follows from

\[ \| u(\cdot + \eta) - u \|_2 \leq C \sqrt{|\eta|} \| u \|_{1,2, M_n} \text{ if } u \in H_{M_n} \]
Discrete setting, Stationary case, M1

\[ \rho_n \to \rho \text{ weakly in } L^2(\Omega) \]
\[ u_n \to u \text{ weakly in } L^2(\Omega) \]
\[ (u_n)_n \text{ is bounded in } H_{M_n}, \| \cdot \|_{1,2,M_n} \]

“Compact embedding” of \( (H_{M_n}, \| \cdot \|_{1,2,M_n})_n \) in \( L^2(\Omega) \)

Then \( \int \rho_n u_n \to \int \rho u \)

Non admissible meshes: Compactness follows from \( (d=3) \)
\[ \| u(\cdot + \eta) - u \|_2 \leq C \| \eta \|^{\frac{2}{5}} \| u \|_{1,2,M_n} \text{ if } u \in H_{M_n} \]

Proof using, for \( u \in H_n, \)
\[ \| u(\cdot + \eta) - u \|_{L^1(\mathbb{R}^3)} \leq |\eta| \sqrt{d} \| u \|_{1,2,n} \text{ and} \]
\[ \| u \|_{L^6(\mathbb{R}^3)} \leq C \| u \|_{1,2,n} \text{ if } u \in H_n \text{ (Discrete Sobolev embedding)} \]
Discrete setting, Stationary case, M2

\[ \rho_n, u_n \in H_{\mathcal{M}_n}, \text{ size}(\mathcal{M}_n) \to 0 \text{ as } n \to \infty \text{ (regularity of the meshes)} \]

\[ \rho_n \to \rho \text{ weakly in } L^2(\Omega) \]
\[ u_n \to u \text{ weakly in } L^2(\Omega) \]
\[ (u_n)_n \text{ is bounded in } H_{\mathcal{M}_n}, \| \cdot \|_{1,2,\mathcal{M}_n} \]

This gives \((u_n)_n\) is bounded in \(H^s(\Omega), 0 < s < 2/5\)

Identify \(L^2(\Omega)\)' with \(L^2(\Omega)\), since \(H^s(\Omega)\) is compact in \(L^2(\Omega)\),
Compact embedding of \(L^2(\Omega)\) in \(H^{-s}(\Omega)\)

Then
\[ u_n \to u \text{ weakly in } H^s(\Omega) \]
\[ \rho_n \to \rho \text{ in } H^{-s}(\Omega) \]

and \[ \int_\Omega \rho_n u_n = \langle \rho_n, u_n \rangle_{H^{-s}, H^s} \to \langle \rho, u \rangle_{H^{-s}, H^s} = \int_\Omega \rho u \]
Continuous setting, evolution case

\( \rho_n \to \rho \) weakly in \( L^2([0, T[, L^2(\Omega)) \)

\( u_n \to u \) weakly in \( L^2([0, T[, L^2(\Omega)) \)

Question : \( \int_{]0, T[ \times \Omega} \rho_n u_n \to \int_{]0, T[ \times \Omega} \rho u ? \)

- in general, no. Even if \((u_n)_n\) is bounded in \( L^2([0, T[, H^1_0(\Omega)) \)
  No compactness of \( L^2([0, T[, H^1_0(\Omega)) \) in \( L^2([0, T[, L^2(\Omega)) \)

- yes if \((u_n)_n\) is bounded in \( H^1([0, T[, H^1_0(\Omega)) \) since
  compactness of \( H^1([0, T[, H^1_0(\Omega)) \) in \( L^2([0, T[, L^2(\Omega)) \)

- yes if \((\rho_n)_n\) is bounded in \( H^1([0, T[, L^2(\Omega)) \) since compactness
  of \( H^1([0, T[, L^2(\Omega)) \) in \( L^2([0, T[, H^{-1}(\Omega)) \)

Is it possible to use weaker hypotheses on \((\partial_t u_n)_n\) or \((\partial_t \rho_n)_n\) ?
Continuous setting, (Generalized) Aubin-Simon Compactness Lemma

$X$, $B$, $Y$ are three Banach spaces, $X \subset B$, $X \subset Y$ such that
1. $X$ compactly embedded in $B$
2. $\|w_n\|_X \leq C$, $\|w_n - w\|_B \to 0$, $\|w_n\|_Y \to 0$ implies $w = 0$

Let $T > 0$ $1 \leq p < +\infty$ and $(u_n)_{n \in \mathbb{N}}$ be a sequence such that

$\Rightarrow (u_n)_{n \in \mathbb{N}}$ is bounded in $L^p(0, T, X)$,
$\Rightarrow (\partial_t u_n)_{n \in \mathbb{N}}$ is bounded in $L^1(0, T, Y)$.

Then there exists $u \in L^p(0, T, B)$ such that, up to a subsequence, $u_n \to u$ in $L^p(0, T, B)$

Particular cases for hypothesis 2:

Easy case : $Y = X$ or $B$ or, more generally, $\| \cdot \|_B \leq C \| \cdot \|_Y$
Aubin Simon : $B$ continuously embedded in $Y$, $\| \cdot \|_Y \leq C \| \cdot \|_B$
Generalized Lions lemma (crucial if \( \| \cdot \|_B \not\leq C \| \cdot \|_Y \))

\( X, B, Y \) are three Banach spaces, \( X \subset B, X \subset Y \) such that
1. \( X \) compactly embedded in \( B \)
2. \( \| w_n \|_X \leq C, \| w_n - w \|_B \to 0, \| w_n \|_Y \to 0 \) implies \( w = 0 \)

Then, for any \( \varepsilon > 0 \), there exists \( C_\varepsilon \) such that, for \( w \in X \),

\[
\| w \|_B \leq \varepsilon \| w \|_X + C_\varepsilon \| w \|_Y.
\]

Proof: By contradiction
Classical Lions lemma, a particular case, simpler

$B$ is a Hilbert space and $X$ is a Banach space $X \subset B$. We define on $X$ the dual norm of $\| \cdot \|_X$, with the scalar product of $B$, namely

$$\|u\|_Y = \sup\{(u|v)_B, \ v \in X, \|v\|_X \leq 1\}.$$  

Then, for any $\varepsilon > 0$ and $w \in X$,

$$\|w\|_B \leq \varepsilon \|w\|_X + \frac{1}{\varepsilon} \|w\|_Y.$$

The proof is simple since

$$\|u\|_B = (u|u)_B^{\frac{1}{2}} \leq (\|u\|_Y \|u\|_X)^{\frac{1}{2}} \leq \varepsilon \|w\|_X + \frac{1}{\varepsilon} \|w\|_Y.$$  

Compactness of $X$ in $B$ is not needed here (but this compactness is needed for Aubin-Simon Compactness Lemma).
Continuous setting, evolution case, compressible NS, M2

\( \rho_n \to \rho \) weakly in \( L^2(]0, T[, L^2(\Omega)) \), if \( \gamma \geq 2 \)

\( u_n \to u \) weakly in \( L^2(]0, T[, L^2(\Omega)^3) \)

\((u_n)_n\) is bounded in \( L^2(]0, T[, H^1_0(\Omega)^3) \)

\( \partial_t \rho_n + \text{div} (\rho_n u_n) = 0 \)

Then \( (\partial_t \rho_n)_n \) is bounded in \( L^1(]0, T[, W^{-1,1}(\Omega)) \)

This gives compactness of \((\rho_n)_n\) in \( L^2(]0, T[, H^{-1}(\Omega)) \)

(Aubin-Simon compactness Theorem with : \( X = L^2(\Omega), B = H^{-1}(\Omega), Y = W^{-1,1}(\Omega) \))

\( u_n \to u \) weakly in \( L^2(]0, T[, H^1_0(\Omega)^3) \)

\( \rho_n \to \rho \) in \( L^2(]0, T[, H^{-1}(\Omega)) \)

and, for any regular \( \varphi \),

\[
\int_{]0, T[ \times \Omega} \rho_n u_n \cdot \nabla \varphi = \langle \rho_n, u_n \cdot \nabla \varphi \rangle_{L^2(H^{-1}), L^2(H^1_0)} \to \int_{]0, T[ \times \Omega} \rho u \cdot \nabla \varphi
\]

which gives \( \partial_t \rho + \text{div}(\rho u) = 0 \)
Continuous setting, evolution case, compressible NS, M2

\( u_n \to u \) weakly in \( L^2(]0, T[, H^1_0(\Omega)^3) \)
\( \rho_n u_n \to \rho u \) weakly in \( L^2(]0, T[, L^r(\Omega)^3) \)
with \( r = \frac{6\gamma}{6 + \gamma} \geq 2 \) if \( \gamma \geq 3 \)

\[ \partial_t (\rho_n u_n) + \text{div}(\rho_n u_n \otimes u_n) - \Delta u_n + \nabla p_n = f \]
Then \( (\partial_t (\rho_n u_n))_n \) is bounded in \( L^1(]0, T[, W^{-1,1}(\Omega)^3) \)

This gives compactness of \( (\rho_n u_n)_n \) in \( L^2(]0, T[, H^{-1}(\Omega)^3) \)
(Aubin-Simon compactness Theorem with :
\( X = L^2(\Omega), B = H^{-1}(\Omega), Y = W^{-1,1}(\Omega) \))

\( u_n \to u \) weakly in \( L^2(]0, T[, H^1_0(\Omega)^3) \)
\( \rho_n u_n \to \rho u \) in \( L^2(]0, T[, H^{-1}(\Omega)^3) \)

Which gives the convergence (in the distributional sense) of
\( \rho_n u_n \otimes u_n \) to \( \rho u \otimes u \) (and allows passing to the limit in the
momentum equation)
Continuous setting, evolution case, Stefan, M1

$$\rho_n \to \rho \text{ weakly in } L^2([0, T], L^2(\Omega))$$
$$u_n \to u \text{ weakly in } L^2([0, T], L^2(\Omega))$$
$$(u_n)_n \text{ is bounded in } L^2([0, T], H^1_0(\Omega))$$
$$\partial_t \rho_n - \Delta u_n = 0, \ u_n = \varphi(\rho_n)$$

$$\varphi \in C(\mathbb{R}, \mathbb{R}) \text{ is nondecreasing } \varphi' = 0 \text{ on } ]a, b[, \ a < b$$

one has $$\partial_t \rho - \Delta u = 0$$, but $$u = \varphi(\rho)$$?

First step : pass to the limit on $$\int \rho_n u_n$$

no direct estimate on $$\partial_t u_n$$, but (Alt-Luckaus trick) estimate on the time-translates of $$u_n$$

Then compactness of $$(u_n)_n \text{ in } L^2([0, T], L^2(\Omega))$$

$$u_n \to u \text{ in } L^2([0, T], L^2(\Omega))$$
$$\rho_n \to \rho \text{ weakly in } L^2([0, T], L^2(\Omega))$$
and, $$\int_{[0, T] \times \Omega} \rho_n u_n \to \int_{[0, T] \times \Omega} \rho u$$

Second step : Minty trick, $$u = \varphi(\rho)$$
Minty trick

\[ \rho_n \to \rho \text{ weakly in } L^2 \quad (L^2 = L^2(\Omega) \text{ or } L^2([0, T], L^2(\Omega))) \]
\[ u_n \to u \text{ weakly in } L^2 \]
\[ \int \rho_n u_n \to \int \rho u \]
\[ u_n = \varphi(\rho_n) \]

\( \varphi \in C(\mathbb{R}, \mathbb{R}) \) is nondecreasing, \( |\varphi(s)| \leq C|s| \)

Question : \( u = \varphi(\rho) \) ? for any \( \bar{\rho} \in L^2 \)

\[ 0 \leq \int (\rho_n - \bar{\rho})(\varphi(\rho_n) - \varphi(\bar{\rho})) = \int (\rho_n - \bar{\rho})(u_n - \varphi(\bar{\rho})) \]

as \( n \to \infty \), \( 0 \leq \int (\rho - \bar{\rho})(u - \varphi(\bar{\rho})) \)

\( \bar{\rho} = \rho - \varepsilon \psi, \varepsilon > 0 \) and \( \psi \) regular function,

\[ 0 \leq \int \psi(u - \varphi(\rho - \varepsilon \psi)) \]

\( \varepsilon \to 0 \), \( \psi \) and \( -\psi \) give \( \int \psi(u - \varphi(\rho)) = 0 \) and then \( u = \varphi(\rho) \)
Continuous setting, evolution case, Stefan, M2

\[ \rho_n \rightarrow \rho \text{ weakly in } L^2([0, T[, L^2(\Omega)) \]
\[ u_n \rightarrow u \text{ weakly in } L^2([0, T[, L^2(\Omega)) \]
\[ (u_n)_n \text{ is bounded in } L^2([0, T[, H^1_0(\Omega)) \]
\[ \partial_t \rho_n - \Delta u_n = 0, \quad u_n = \varphi(\rho_n) \]

Then \( (\partial_t \rho_n)_n \) bounded in \( L^2([0, T[, H^{-1}(\Omega)) \)
This gives compactness of \( (\rho_n)_n \) in \( L^2([0, T[, H^{-1}(\Omega)) \)
(Aubin-Simon Theorem with : \( X = L^2(\Omega), \ B = Y = H^{-1}(\Omega) \))

\[ u_n \rightarrow u \text{ weakly in } L^2([0, T[, H^1_0(\Omega)) \]
\[ \rho_n \rightarrow \rho \text{ in } L^2([0, T[, H^{-1}(\Omega)) \]

and, \[ \int_{[0, T[ \times \Omega} \rho_n u_n \rightarrow \int_{[0, T[ \times \Omega} \rho u \]

which gives (Minty trick) \( u = \varphi(\rho) \)
Use of the compactness lemma in the previous examples

For compressible Navier Stokes eqs:
\( B = H^{-1}(\Omega), \ X = L^2(\Omega), \ Y = W^{-1,1}(\Omega) \)

For Stefan problem:
\( X = L^2(\Omega), \ B = Y = H^{-1}(\Omega) \)

Is it possible to have discrete versions of these compactness results, for proving the convergence of numerical schemes?
Space-Time discretization

$T > 0$, time step $k = \frac{T}{N}$

- $H_M$ the space of functions from $\Omega$ to $\mathbb{R}$, constant on each $K$, $K \in M$.
- The function $u$ is constant on $K \times ((p - 1)k, pk)$ with $K \in M$ and $p \in \{1, \ldots, N\}$.
  
  $u(\cdot, t) = u(p)$ for $t \in ((p - 1)k, pk)$ and $u(p) \in H_M$.

- Discrete derivatives in time, $\partial_{t,k} u$, defined by:

  $$\partial_{t,k} u(\cdot, t) = \partial_{t,k}^{(p)} u = \frac{1}{k} (u(p) - u(p-1)) \text{ for } t \in ((p - 1)k, pk),$$

  for $p \in \{2, \ldots, N\}$ (and $\partial_{t,k} u(\cdot, t) = 0$ for $t \in (0, k)$).

$M$ can be different for $\rho$, $p$ and each component of the velocity (MAC-scheme)
Discrete Lions lemma

$B$ is a Banach space, $(B_n)_{n \in \mathbb{N}}$ is a sequence of finite dimensional subspaces of $B$. $\| \cdot \|_X$ and $\| \cdot \|_Y$ are two norms on $B_n$ such that:

If $(\|w_n\|_X)_{n \in \mathbb{N}}$ is bounded, then,

- up to a subsequence, there exists $w \in B$ s.t. $w_n \to w$ in $B$.
- If $\|w_n - w\|_B \to 0$ and $\|w_n\|_Y \to 0$, then $w = 0$.

Then, for any $\varepsilon > 0$, there exists $C_\varepsilon$ such that, for $n \in \mathbb{N}$ and $w \in B_n$

$$\|w\|_B \leq \varepsilon \|w\|_X + C_\varepsilon \|w\|_Y.$$  

Example: $B_n = H_{\mathcal{M}_n}$ (the finite dimensional space given by the mesh $\mathcal{M}_n$). We have to choose $B$, $\| \cdot \|_X$ and $\| \cdot \|_Y$.  

Discrete Lions lemma, proof

Proof by contradiction. There exists \( \varepsilon > 0 \) and \((w_n)_{n \in \mathbb{N}}\) such that, for all \( n, \ w_n \in B_n \) and

\[
\|w_n\|_B > \varepsilon \|w_n\|_X + C_n \|w_n\|_Y,
\]

with \( \lim_{n \to \infty} C_n = +\infty \).

It is possible to assume that \( \|w_n\|_B = 1 \). Then \((\|w_n\|_X)_{n \in \mathbb{N}}\) is bounded and, up to a subsequence, \( w_n \to w \) in \( B \) (so that \( \|w\|_B = 1 \)). But \( \|w_n\|_Y \to 0 \), so that \( w = 0 \), in contradiction with \( \|w\|_B = 1 \).
Discrete Compactness Lemma

$B$ a Banach, $1 \leq p < +\infty$, $(B_n)_{n \in \mathbb{N}}$ family of finite dimensional subspaces of $B$. $\| \cdot \|_{X_n}$ and $\| \cdot \|_{Y_n}$ two norms on $B_n$ such that:

If $(\| w_n \|_{X_n})_{n \in \mathbb{N}}$ is bounded, then,

- up to a subsequence, there exists $w \in B$ s.t. $w_n \to w$ in $B$.
- If $\| w_n - w \|_{B} \to 0$ and $\| w_n \|_{Y_n} \to 0$, then $w = 0$.

$X_n = B_n$ with norm $\| \cdot \|_{X_n}$, $Y_n = B_n$ with norm $\| \cdot \|_{Y_n}$. Let $T > 0$, $k_n > 0$ and $(u_n)_{n \in \mathbb{N}}$ be a sequence such that

- for all $n$, $u_n(\cdot, t) = u_n^{(p)} \in B_n$ for $t \in ((p-1)k_n, pk_n)$
- $(u_n)_{n \in \mathbb{N}}$ is bounded in $L^p((0, T), X_n)$,
- $(\partial_{t,k_n} u_n)_{n \in \mathbb{N}}$ is bounded in $L^1((0, T), Y_n)$.

Then there exists $u \in L^p((0, T), B)$ such that, up to a subsequence, $u_n \to u$ in $L^p((0, T), B)$.

Example: $B_n = H_{\mathcal{M}_n}$. We have to choose $B$, $\| \cdot \|_{X_n}$, $\| \cdot \|_{Y_n}$
Discrete setting, evolution case, compressible NS, M2

\( \rho_n \to \rho \) weakly in \( L^2(]0, T[, L^2(\Omega)) \) \((\gamma \geq 2)\)

\( u_n \to u \) weakly in \( L^2(]0, T[, L^2(\Omega)^3) \)

\((u_n)_n\) is bounded in \( L^2(]0, T[, H^s_n) \), with \( \| \cdot \|_{1,2,M^{(i)}_n} \)

\( \partial_t, k_n \rho_n + \text{div}_n(\rho_n u_n) = 0 \)

Then \((\partial_t, k_n \rho_n)_n\) is bounded in \( L^1(]0, T[, Y_n) \)

where \( Y_n = H^{(i)}_n \) with :

\[
\| w \|_{Y_n} = \max \{ \int w \varphi; \, \varphi \in H^{(i)}_n; \| \nabla_n \varphi \|_{L^\infty(\Omega)} + \| \varphi \|_{L^\infty(\Omega)} = 1 \}.
\]

Compactness Theorem with

\( B = H^{-s}(\Omega) \) and \( X_n = H^{(i)}_n \) with \( L^2(\Omega) \)-norm

gives compactness of \((\rho_n)_n\) in \( L^2(]0, T[, H^{-s}(\Omega)) \), \(0 < s < 1/2\)

\( u_n \to u \) weakly in \( L^2(]0, T[, H^s(\Omega)^3) \)

\( \rho_n \to \rho \) in \( L^2(]0, T[, H^{-s}(\Omega)) \)

and, for any regular \( \varphi \),

\[
\int \rho_n u_n \cdot \nabla \mathcal{M}_n \varphi = \langle \rho_n, u_n \cdot \nabla \varphi \rangle_{L^2(H^{-s}),L^2(H^s)} + R \to \int \rho u \cdot \nabla \varphi
\]
Discrete setting, evolution case, compressible NS, M2

Similarly it is possible to prove the convergence of \( \text{div}_n \rho_n u_n \otimes u_n \) to \( \text{div} \rho u \otimes u \)

\[
\rho_n u_n \to \rho u \text{ weakly in } L^2([0, T[, L^2(\Omega)^3) \text{ (if } \gamma \geq 3) \\
u_n \to u \text{ weakly in } L^2([0, T[, L^2(\Omega)^3) \\
(u_n)_n \text{ is bounded in } L^2([0, T[, H_n) \text{, with } || \cdot ||_{1,2, M_n^{(i)}}
\]

Using the discrete momentum equation, one has essentially (for each component of \( u_n \))

\[
(\partial_{t,k_n}(\rho_n u_n))_n \text{ is bounded in } L^1([0, T[, Y_n) \\
\text{where } Y_n = H_n^{(i)} \text{ (mesh for a component of } u_n) \text{ with :}
\]

\[
|| w ||_{Y_n} = \max\{ \int_\Omega w \varphi; \varphi \in H_n^{(i)}; || \nabla_n \varphi ||_{L^\infty(\Omega)} + || \varphi ||_{L^\infty(\Omega)} = 1 \}.
\]

Compactness Theorem with \( B = H^{-s}(\Omega) \) and \( X_n = H_n^{(i)} \) with \( L^2(\Omega) \)-norm

\[
gives compactness of \( (\rho_n u_n)_n \text{ in } L^2([0, T[, H^{-s}(\Omega)^3) \), \( 0 < s < 1/2 \)

which allows to prove \( \partial_t \rho u + \text{div} (\rho u \otimes u) - \Delta u + \nabla p = f \)
Discrete setting, evolution case, Stefan, M1

\( \rho_n \to \rho \) weakly in \( L^2(]0, T[, L^2(\Omega)) \)

\( u_n \to u \) weakly in \( L^2(]0, T[, L^2(\Omega)) \)

\((u_n)_n\) is bounded in \( L^2(]0, T[, H_{\mathcal{M}_n}(\Omega)) \) with \( \| \cdot \|_{1,2,\mathcal{M}_n} \)

\( \partial_{t,k_n}\rho_n - \Delta_n u_n = 0, \ u_n = \varphi(\rho_n) \)

\( \varphi \in C(\mathbb{R}, \mathbb{R}) \) is nondecreasing \( \varphi' = 0 \) on \( ]a, b[\), \( a < b \)

one has \( \partial_t \rho - \Delta u = 0 \), but \( u = \varphi(\rho) \) ?

First step: pass to the limit on \( \int \rho_n u_n \)

no direct estimate on \( \partial_{t,k_n}u_n \), but a discrete version of Alt-Luckaus trick gives an estimate on the time-translates of \( u_n \)

Then compactness of \( (u_n)_n \) in \( L^2(]0, T[, L^2(\Omega)) \)

\( u_n \to u \) in \( L^2(]0, T[, L^2(\Omega)) \)

\( \rho_n \to \rho \) weakly in \( L^2(]0, T[, L^2(\Omega)) \)

and, \( \int_{]0, T[\times\Omega} \rho_n u_n \to \int_{]0, T[\times\Omega} \rho u \)

Second step: Minty trick, \( u = \varphi(\rho) \)
Discrete setting, evolution case, Stefan, M2

\[ \rho_n \to \rho \text{ weakly in } L^2([0, T[, L^2(\Omega)) \]
\[ u_n \to u \text{ weakly in } L^2([0, T[, L^2(\Omega)) \]

\((u_n)_n\) is bounded in \( L^2([0, T[, H_{M_n}) \) with \( \| \cdot \|_{1,2, M_n} \)

\[ \partial_{t,k_n} \rho_n - \Delta M_n u_n = 0, \quad u_n = \varphi(\rho_n) \]

First step: pass to the limit on \( \int \rho_n u_n \)

\((\partial_{t,k_n} \rho_n)_n\) bounded in \( L^2([0, T[, H_{M_n}) \) with \( \| \cdot \|_{-1,2, M_n} \)

This gives compactness of \((\rho_n)_n\) in \( L^2([0, T[, H^{-s}(\Omega)) \)

\[ B = H^{-s}(\Omega), \quad B_n = H_{M_n}, \quad \| \cdot \|_{X_n} = \| \cdot \|_{L^2(\Omega)}, \]
\[ \| \cdot \|_{Y_n} = \| \cdot \|_{-1,2, M_n} \text{ (the dual norm of the norm } \| \cdot \|_{1,2, M_n} \) \]

\[ \rho_n \to \rho \text{ in } L^2([0, T[, H^{-s}(\Omega)) (0 < s < 1/2) \]
\[ u_n \to u \text{ weakly in } L^2([0, T[, H^s(\Omega)) \]

and, \( \int_{[0,T] \times \Omega} \rho_n u_n \to \int_{[0,T] \times \Omega} \rho u \)

Second step: Minty trick, \( u = \varphi(\rho) \)
Spaces $B$, $X_n$, $Y_n$ for compressible NS

$B = H^{-s}(\Omega)$, $0 < s < 1/2$

$Y_n = H_{\mathcal{M}_n}$ with $\| \cdot \|_{-1,1,\mathcal{M}_n}$

$X_n = H_{\mathcal{M}_n}$ with $L^2(\Omega)$-norm

- Compact embedding of $L^2(\Omega)$ in $H^{-s}(\Omega)$
- If $w_n \in H_{\mathcal{M}_n}$, $w_n \rightharpoonup w$ weakly in $L^2(\Omega)$ and $\|w_n\|_{-1,1,\mathcal{M}_n} \to 0$, then $w = 0$? Yes... Proof:

Let $\varphi \in W^{1,\infty}_0(\Omega)$ and its “projection” $\pi_n \varphi \in H_{\mathcal{M}_n}$. One has

$\|\pi_n \varphi\|_{1,\infty,\mathcal{M}_n} \leq \|\varphi\|_{W^{1,\infty}(\Omega)}$ and then

$$\left| \int_{\Omega} w_n(\pi_n \varphi) dx \right| \leq \|w_n\|_{-1,1,\mathcal{M}_n} \|\varphi\|_{W^{1,\infty}(\Omega)} \to 0,$$

and, since $w_n \rightharpoonup w$ weakly in $L^1(\Omega)$ and $\pi_n \varphi \to \varphi$ uniformly,

$$\int_{\Omega} w_n(\pi_n \varphi) dx \to \int_{\Omega} w \varphi dx.$$

This gives $\int_{\Omega} w \varphi dx = 0$ for all $\varphi \in W^{1,\infty}_0(\Omega)$ and then $w = 0$ a.e.