The pseudo-effective cone of a non-Kählerian surface and applications

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June 30, 2007

Abstract

We describe the positive cone and the pseudo-effective cone of a non-Kählerian surface. We use these results for two types of applications:

1. Describe the set $\sigma(X) \subset \mathbb{R}$ of possible total Ricci scalars associated with Gauduchon metrics of fixed volume 1 on a fixed non-Kählerian surface, and decide whether the assignment $X \mapsto \sigma(X)$ is a deformation invariant.
2. Study the stability of the canonical extension

$$0 \to \mathcal{K}_X \to \mathcal{A} \to \mathcal{O}_X \to 0$$

of a class VII surface X with positive b_2. This extension plays an important role in our strategy to prove the GSS conjecture using gauge theoretical methods [Te2], [Te3].

Our main tools are Buchdahl's ampleness criterion for non-Kählerian surfaces [Bu2] and the recent results of Dloussky-Oeljeklaus-Toma [DOT] and Dloussky [D] on class VII surfaces with curves.

MSC: 32J15, 32Q57, 32L05, 32G13

1 Introduction

In this paper we study certain complex geometric and differential geometric properties of non-Kählerian surfaces. The first problem we treat is the following:

Describe explicitly the pseudo-effective cone of a non-Kählerian surface and compare it with the effective cone.

By definition, the pseudo-effective cone of a complex surface is the subset of the Bott-Chern cohomology space $H^{1,1}_{BC}(X,\mathbb{R})$ consisting of $i\bar\partial\partial$-classes which are represented by closed positive currents. The effective cone is just the cone generated by classes associated with effective divisors.

We will solve completely this problem showing that the pseudo-effective cone is determined in a simple way by the finite set of irreducible effective divisors with negative self-intersection. The proof is based on a version of Buchdahl's ampleness criterion [Bu2], which will be explained in the first section. This criterion will provide a simple description of the *positive cone* of a non-Kählerian surface, i.e. the cone of $i\bar\partial\partial$-closed (1,1)-classes (modulo $i\bar\partial\partial$-exact forms) associated with Gauduchon metrics [G].

We point out that *all our results do not make use of the GSS conjecture; in particular they hold for the still non-classified class VII-surfaces with second Betti number $b_2 > 1$.*

Our description of the pseudo-effective cone will allow us to solve the following two problems.

1. *Determine the possible values of the total Ricci scalars of the Gauduchon metrics with volume 1 on a given non-Kählerian surface.*

   For a Hermitian metric g we denote by s_g the *Ricci scalar* of g (see [G]), which is defined by the formula

$$s_g := i\Lambda_g \mathrm{Tr}(F_{A_g}) \ ,$$

   where A_g is the Chern connection associated with g and the holomorphic structure on the tangent bundle. In the non-Kählerian framework, s_g does not coincide in general with the scalar curvature of the Riemannian metric g. The total Ricci scalar of g is defined by

$$\sigma_g := \int_X s_g vol_g = \int_X i\mathrm{Tr}(F_{A_g}) \wedge \omega_g \ .$$

   Let $\mathcal{G}(X)$ be the set of Gauduchon metrics of X. Our problem is to determine the set

$$\sigma(X) := \{\sigma_g |\ g \in \mathcal{G}(X),\ \int_X vol_g = 1\}.$$

   We will see for instance that, for a (blown up) Inoue surface X, one has $\sigma(X) = (-\infty, 0)$, which might be surprising for a surface with $\mathrm{kod}(X) = -\infty$. We will also answer the following natural question:

   Is the assignment $X \mapsto \sigma(X) \subset \mathbb{R}$ a deformation invariant ?

   Using certain families of class VII-surfaces, we will see that the answer is in general negative.

2. *The stability of the canonical extension of a class VII-surface.*

   Let X be a class VII-surface (i.e. a surface with $b_1(X) = 1$ and $\mathrm{kod}(X) = -\infty$). Such a surface has $h^1(X, \mathcal{O}_X) = 1$ so, by Serre duality, one also has

$h^1(X, \mathcal{K}_X) = 1$. The *canonical extension* of X is defined to be the unique (up to the natural $\mathbb{C}^*$-action on $\mathrm{Ext}^1(\mathcal{O}_X, \mathcal{K}_X)$) nontrivial extension of the form

$$0 \longrightarrow \mathcal{K}_X \longrightarrow \mathcal{A} \longrightarrow \mathcal{O}_X \longrightarrow 0 \tag{1}$$

The question here is whether there exists Gauduchon metrics on X with respect to which $\mathcal{A}$ is stable.

We will see that, excepting certain very special surfaces with global spherical shell, every minimal class VII surface with positive b_2 admits Gauduchon metrics g for which the bundle $\mathcal{A}$ is g-stable. The motivation for this problem is the following:

For any topologically trivial line bundle $\mathcal{L} \in \mathrm{Pic}^0(X) \simeq \mathbb{C}^*$ with $\mathcal{L}^{\otimes 2} \not\simeq \mathcal{O}_X$ one has $\mathrm{Ext}^1(\mathcal{L}, \mathcal{K}_X \otimes \mathcal{L}^{-1}) = 0$, so there are no non-trivial extensions of $\mathcal{L}$ by $\mathcal{K}_X \otimes \mathcal{L}^{-1}$. On the other hand, the dimension of the moduli space $\mathcal{M}^{\mathrm{st}}(0, \mathcal{K}_X)$ of stable rank 2-bundles $\mathcal{E}$ with $c_2(\mathcal{E}) = 0$ and $\det(\mathcal{E}) = \mathcal{K}_X$ is $b_2(X)$.

Therefore, although the extension (1) is rigid (it cannot be deformed in another extension of the form $0 \to \mathcal{K}_X \otimes \mathcal{L}^{-1} \to \mathcal{E} \to \mathcal{L} \to 0$ with $[\mathcal{L}] \in \mathrm{Pic}^0(X)$), its central term $\mathcal{A}$ cannot be rigid for $b_2(X) > 0$.

As in [Te2], using the Kobayashi-Hitchin correspondence on non-Kählerian surfaces ([Bu1], [LY], [LT]) one can prove that, if X had no curve and $b_2(X) \leq 3$, the connected component of $[\mathcal{A}]$ in $\mathcal{M}^{\mathrm{st}}(0, \mathcal{K}_X)$ would be a smooth compact manifold containing both filtrable and non-filtrable points. This is the starting point of our strategy to prove the GSS conjecture using gauge theoretical methods.

2 Buchdahl's ampleness criterion and positiveness

In [Bu2] Buchdahl proved an interesting ampleness criterion for (non-algebraic) complex surfaces; surprisingly, his statement is very much similar to the algebraic geometric Nakai-Moishezon criterion. This result suggests that certain fundamental purely algebraic geometric theorems might have natural extensions to the non-algebraic and even non-Kählerian framework; the difficulty is to find the correct complex geometric analogues of the algebraic geometric notions involved in the original statement.

Theorem 2.1 *[Bu2] Let X be a compact complex surface equipped with a positive $\bar\partial\partial$-closed (1,1)-form ω_0 and let φ be a smooth real $\bar\partial\partial$-closed (1,1)-form satisfying*

1. $\int_X \varphi \wedge \varphi > 0$,
2. $\int_X \varphi \wedge \omega_0 > 0$,

3. $\int_D \varphi > 0$ *for every irreducible effective divisor with* $D^2 < 0$.

Then there is a smooth function ψ *on* X *such that* $\varphi + i\bar{\partial}\partial\psi$ *is positive.*

This result shows that it is very natural to extend the fundamental algebraic geometric notion "positive cone'" to the non-algebraic non-Kählerian framework in the following way:

Set

$$Q_0 := i\bar{\partial}\partial : A^0(X,\mathbb{R}) \longrightarrow A^{1,1}(X,\mathbb{R}) \ , \ Q_1 := i\bar{\partial}\partial : A^{1,1}(X,\mathbb{R}) \longrightarrow A^{2,2}(X,\mathbb{R}) \ ,$$

$$\mathcal{H}(X) := {}^{\ker(Q_1)}\big/_{\mathrm{im}(Q_0)} \ .$$

It is easy to see that $\mathrm{im}(Q_0)$ is closed: it suffices to choose a Hermitian metric g on X and to note that the operator $P_g := \Lambda_g \circ Q_0$ is elliptic. Therefore $\mathcal{H}(X)$ is a Fréchet space. It is not difficult to see that this space is infinite dimensional; it contains the finite dimensional Bott-Chern cohomology space (see [BHPV], p. 148)

$$H^{1,1}_{BC}(X,\mathbb{R}) := {}^{\ker(d : A^{1,1}(X,\mathbb{R}) \longrightarrow A^3(X,\mathbb{R}))}\big/_{i\bar{\partial}\partial(A^0(X,\mathbb{R}))} \ .$$

Definition 2.2 *Let* X *be a compact complex surface and* $\mathcal{G}(X)$ *the space of Gauduchon metrics on* X*. The positive cone of* X *is the open cone* $\mathcal{H}_+(X) \subset \mathcal{H}(X)$ *defined by*

$$\mathcal{H}_+(X) := \{[\omega_g] |\ g \in \mathcal{G}(X)\} \ .$$

Note that one has a natural well defined intersection form

$$\mathcal{H}(X) \times \mathcal{H}(X) \to \mathbb{R}$$

given by $[\eta]\cdot[\mu] \mapsto \int_X \eta \wedge \mu$. Moreover, every real $i\bar{\partial}\partial$-closed $(1,1)$-current u defines a linear form $\langle \cdot, u\rangle : \mathcal{H}(X) \to \mathbb{R}$.

Buchdahl's criterion says that the elements h of the positive cone $\mathcal{H}_+$ are characterized by the system of inequalities:

1. $h^2 > 0$,
2. $h \cdot [\omega_0] > 0$,
3. $\langle h, [D]\rangle > 0$ for every irreducible effective divisor D with $D^2 < 0$.

In the non-Kählerian case ($b_1(X)$ odd), one can reformulate this criterion, by replacing the class $[\omega_0]$ in the second inequality with the class of an *exact* form. This modification, which is explained in detail below, is very useful, because all linear inequalities in the resulting system will be associated with classes in the Bott-Chern cohomology space $H^{1,1}_{BC}(X,\mathbb{R})$.

For a complex surface X we put

$$B^{1,1}(X,\mathbb{R}) := d(A^1(X,\mathbb{R})) \cap A^{1,1}(X,\mathbb{R}) \supset i\bar\partial\partial(A^0(X,\mathbb{R}))\ ,$$

$$H^{1,1}(X,\mathbb{R}) := {}^{\ker(d:A^{1,1}(X,\mathbb{R})\longrightarrow A^3(X,\mathbb{R}))}\big/_{B^{1,1}(X,\mathbb{R})} \subset H^2(X,\mathbb{R})\ .$$

Some of the statements in the following lemma are probably known. We include short proofs for completeness.

Lemma 2.3 *Let g be a Gauduchon metric on X.*

1. *One has the exact sequences*

$$0 \longrightarrow {}^{B^{1,1}(X,\mathbb{R})}\big/_{i\bar\partial\partial(A^0(X,\mathbb{R}))} \longrightarrow H^{1,1}_{BC}(X,\mathbb{R}) \longrightarrow H^{1,1}(X,\mathbb{R}) \longrightarrow 0$$

$$0 \longrightarrow i\bar\partial\partial(A^0(X,\mathbb{R})) \longrightarrow B^{1,1}(X,\mathbb{R}) \xrightarrow{\langle\cdot\ ,\ \omega_g\rangle} \mathbb{R}$$

2. *The canonical map $H^1(X,i\mathbb{R}) \to H^1(X,\mathcal{O})$ is injective.*

3. *The following conditions are equivalent*
   - *(a) $\langle\cdot,\omega_g\rangle$ vanishes identically on $B^{1,1}(X,\mathbb{R})$.*
   - *(b) $H^{1,1}_{BC}(X,\mathbb{R}) = H^{1,1}(X,\mathbb{R})$*
   - *(c) The natural monomorphism $H^1(X,i\mathbb{R}) \to H^1(X,\mathcal{O}_X)$ is surjective.*
   - *(d) $b_1(X)$ is even.*

4. *When $b_1(X)$ is odd, one has an exact sequence*

$$0 \longrightarrow \Gamma(X) \longrightarrow H^{1,1}_{BC}(X,\mathbb{R}) \longrightarrow H^{1,1}(X,\mathbb{R}) \longrightarrow 0$$

where $\Gamma := {}^{B^{1,1}(X,\mathbb{R})}\big/_{i\bar\partial\partial(A^0(X,\mathbb{R}))}$ is a line which is identified with $\mathbb{R}$ via $\langle\cdot,\omega_g\rangle$.

Proof: 1. The first exact sequence is obvious. For the second, let $\alpha \in B^{1,1}(X,\mathbb{R})$ such that

$$\int_X \alpha\wedge\omega = 0$$

The Laplace equation $i\Lambda_g\bar\partial\partial u = \Lambda_g\alpha = 0$ is solvable, because the image of the elliptic operator $P_g := i\Lambda_g\bar\partial\partial$ is precisely $\ker\langle\cdot,\omega_g\rangle$; let u_0 be a solution of this equation. The form $i\bar\partial\partial u_0 - \alpha$ is exact and anti-selfdual, so it vanishes.

2. Let $\alpha = -a^{1,0} + a^{0,1}$ be a closed imaginary form (where $a^{0,1} = \overline{a^{1,0}}$) such that $a^{0,1}$ is $\bar\partial$-exact, and let $u \in A^0(X,\mathbb{C})$ such that $\bar\partial u = a^{0,1}$. This implies $\partial\bar u = a^{1,0}$. Since α is closed, we get

$$\partial\bar\partial u - \bar\partial\partial\bar u = -2\bar\partial\partial(\mathrm{Re}(u)) = 0\ ,$$

hence, we can suppose that u is purely imaginary. Then one gets immediately $(\partial+\bar\partial)u=\alpha$.

3. By the second exact sequence in 1., the statements (a) and (b) are both equivalent to the equality

$$\bar\partial\partial(A^0(X,\mathbb{R}))=B^{1,1}(X,\mathbb{R}) \tag{2}$$

To prove $(b)\Rightarrow(c)$ it suffices to show that (2) implies the surjectivity of the map $H^1(X,i\mathbb{R})\to H^1(X,\mathcal{O}_X)$. Let $[\beta^{0,1}]\in H^1(X,\mathcal{O}_X)$. It suffices to find $\varphi\in A^0(X,\mathbb{C})$ such that the form $\alpha^{0,1}:=\beta^{0,1}+\bar\partial\varphi$ satisfies

$$\partial\alpha^{0,1}-\bar\partial\alpha^{1,0}=0$$

(where $\alpha^{1,0}:=\bar\alpha^{0,1}$). Indeed, if we find such a function φ, the de Rham class of $\alpha:=\alpha^{0,1}-\alpha^{1,0}$ will be a preimage of the Dolbeault class of β under the natural morphism $H^1(X,i\mathbb{R})\to H^1(X,\mathcal{O}_X)$. This equation becomes

$$\partial(\beta^{0,1}+\bar\partial\varphi)-\bar\partial(\beta^{1,0}+\partial\bar\varphi)=0\ .$$

We will show that there exists a real solution of this equation. For a real function φ the equation becomes

$$2i\partial\bar\partial\varphi=i(\bar\partial\beta^{1,0}-\partial\beta^{0,1})$$

The form $i(\bar\partial\beta^{1,0}-\partial\beta^{0,1})$ can be written as $d(i\beta^{1,0}-i\beta^{0,1})$ (because $\bar\partial\beta^{1,0}=\partial\beta^{10}=0$), hence it is an exact $(1,1)$ form. Therefore, by hypothesis, it belongs to the image of the operator $i\partial\bar\partial$. The implication $(c)\Rightarrow(d)$ is obvious.

The implication $(d)\Rightarrow(b)$ is well known: it follows from Theorems 2.8, 2.10 and Corollary 13.8 in [BHPV].

4. This follows directly from 1 and 3. ■

Remark 2.4 *Let X be a complex surface with $b_1(X)$ odd. Since the space of Gauduchon metrics on X is connected, the orientation of the "exact line" $\Gamma(X)\subset H^{1,1}_{BC}(X,\mathbb{R})$ induced by $\langle\cdot,\omega_g\rangle$ is well defined. Let γ_0 be a positive generator of this line. One has*

$$\langle\gamma_0\cdot[\omega_g]\rangle>0 \tag{3}$$

for every Gauduchon metric g on X.

Recall now that the Bott-Chern cohomology space $H^{1,1}_{BC}(X,\mathbb{R})$ can be also introduced using currents (see [BHPV], p. 148):

$$H^{1,1}_{BC}(X,\mathbb{R}):={}^{\ker(d:A^{1,1}(X,\mathbb{R})\longrightarrow A^3(X,\mathbb{R}))}\big/_{i\bar\partial\partial(A^0(X,\mathbb{R}))}=$$

$$^{\ker(d:\mathcal{D}'_{1,1}(X,\mathbb{R})\longrightarrow\mathcal{D}'_0(X,\mathbb{R}))}\big/_{i\bar\partial\partial(\mathcal{D}'_{2,2}(X,\mathbb{R}))}\ .$$

Proposition 2.5 *Let X be a complex surface with odd $b_1(X)$. Then the Bott-Chern cohomology class γ_0 is represented by an exact positive current.*

Proof: It is well known that a complex surface with odd first Betti number admits a non-trivial exact positive (1,1)-current (see [Bu2], [La]). This is a refinement – valid for surfaces – of the general Harvey-Lawson's characterization of non-Kählerianity [HL].

Let v be non-trivial exact positive (1,1)-current on X. Then one has obviously $\langle v, \omega_g \rangle > 0$, for every Gauduchon metric on X, so the Bott-Chern cohomology class $[v]$ is a positive multiple of the generator γ_0. ■

We claim that, when $b_1(X)$ is odd, Buchdahl's ampleness criterion is equivalent to the following:

Theorem 2.6 *Let X be a surface with $b_1(X)$ odd. The elements h of the positive cone $\mathcal{H}_+(X)$ are characterized by the inequalities*

1. $h^2 > 0$,
2. $h \cdot \gamma_0 > 0$,
3. $\langle h, D \rangle > 0$ *for every irreducible effective divisor with* $D^2 < 0$.

Proof: Let g_0 be a Gauduchon metric g_0 on X and ω_0 the corresponding form. For any $t \geq 0$, the class $[\omega_0] + t\gamma_0$ satisfies the three inequalities in Buchdahl's criterion, because

$$([\omega_0] + t\gamma_0)^2 = [\omega_0]^2 + 2t\gamma_0 \cdot [\omega_0] \ , \ ([\omega_0] + t\gamma_0) \cdot [\omega_0] = [\omega_0]^2 + t\gamma_0 \cdot [\omega_0] \ ,$$

$$\langle [\omega_0] + t\gamma_0, D \rangle = \langle [\omega_0], D \rangle \ ,$$

for every effective divisor D. Therefore $[\omega_0] + t\gamma_0$ is still the class of the Kähler form, say ω_t, of a Gauduchon metric g_t on X.

Let $h \in \mathcal{H}$ be a class satisfying the three inequalities in the hypothesis. For sufficiently large $t > 0$, one has $h \cdot ([\omega_0] + t\gamma_0) > 0$. Therefore h satisfies Buchdahl's original criterion (for ω_t instead of ω_0). ■

3 Effectiveness and pseudo-effectiveness

Let $\mathcal{D}'_{1,1}(X, \mathbb{R})$, $\mathcal{D}'_0(X, \mathbb{R})$, $\mathcal{D}'_{2,2}(X, \mathbb{R})$ be the dual spaces of $A^{1,1}(X, \mathbb{R})$, $A^0(X, \mathbb{R})$, $A^{2,2}(X, \mathbb{R})$ respectively. Using again the ellipticity of the operator $P_g = i\Lambda_g \bar{\partial}\partial$ associated with a Hermitian metric, one gets easily

Remark 3.1 *The image $Q_1^*(\mathcal{D}'_{2,2}(X, \mathbb{R})) \subset \mathcal{D}'_{1,1}(X, \mathbb{R})$ is closed in $\mathcal{D}'_{1,1}(X, \mathbb{R})$ with respect to the weak topology.*

Corollary 3.2 *The dual space of the quotient $\mathcal{D}'_{1,1}(X) / Q_1^*(\mathcal{D}'_{2,2}(X))$ is naturally isomorphic to* $\ker Q_1$.

Proof: Indeed, by a well-known result about duality in locally convex spaces, the dual space of ${\mathcal{D}'_{1,1}(X)}/{Q_1^*(\mathcal{D}'_{2,2}(X))}$ is just the subspace of $\mathcal{D}'_{1,1}(X)^* = A^{1,1}(X)$ consisting of functionals which vanish on $Q_1^*(\mathcal{D}'_{2,2}(X))$. It's easy to see that this subspace is just $\ker Q_1$. ∎

Remark 3.3 *Let X be complex surface with $b_1(X)$ odd. The set $\mathcal{D}ou(X)^{\mathrm{irr}}_-$ of irreducible effective divisors with negative self-intersection is finite.*

Proof: Indeed, a complex surface X with odd $b_1(X)$ is either an elliptic fibration, or a class VII surface. In the first case, X cannot contain horizontal divisors (because otherwise X would be algebraic) and, on the other hand, the generic fibre has vanishing self-intersection. Therefore, the irreducible effective divisors with negative self-intersection must be components of the (finitely many) singular fibres.

If X is not an elliptic fibration, it must be a class VII surface of vanishing algebraic dimension, so it has only finitely many irreducible effective divisors. ∎

For a finite subset $A \subset E$ of a real vector space E, we denote by $[A]$ the convex hull of A and by CA the cone over A

$$CA := \left\{ \sum_{a \in A} t_a a |\ t_a \geq 0\ \forall a \in A \right\} .$$

Lemma 3.4 *Let E be a locally convex space, and A, $B \subset E$ non-empty finite sets, such that $0 \notin [A] \cup [B]$. Then the following conditions are equivalent:*

1. *$CA \cap CB = \{0\}$,*
2. *There exists a continuous linear form $u \in E^*$ such that*

$$u|_A < 0 \text{ and } u|_B > 0 .$$

Proof: The implication 2. ⇒ 1. is obvious. For the other implication, apply the standard separation theorem in locally convex spaces to the closed convex set CA and the compact convex set $[B]$. These sets are disjoint because, since $0 \notin [B]$, any intersection point would be a non-zero element in $CA \cap CB$. We get a continuous linear form $v \in E^*$ such that

$$\sup_{x \in CA} v(x) < \inf_{y \in [B]} v(y)$$

Since $0 \in CA$, we get $\sup_{x \in CA} v(x) \geq 0$. On the other hand one must have $v(a) \leq 0$ for every $a \in A$ because, otherwise, one would obviously have

$$\sup_{x \in CA} v(x) = +\infty .$$

Therefore $\sup_{x \in CA} v(x) = 0$ and $\inf_{y \in [B]} v(y) > 0$. This means

$$v\big|_A \leq 0 \text{ and } v\big|_B > 0 \ .$$

Since $0 \notin [A]$ and $[A]$ is compact convex set, there exists a continuous linear functional $w \in E^*$ such that $w\big|_A > 0$. Setting $u := v + \varepsilon w$ for sufficiently small $\varepsilon > 0$, we get the desired inequalities. ■

Theorem 3.5 *Let X be complex surface with $b_1(X)$ odd. Let $c \in H^{1,1}_{BC}(X, \mathbb{R})$ be a Chern-Bott cohomology class such that $\langle c, \omega_g \rangle \geq 0$ for every Gauduchon metric g on X. Then $c \in C(\{\gamma_0\} \cup \mathcal{D}ou(X)^{\mathrm{irr}}_-)$.*

Proof: Suppose $c \neq 0$. If the claim was not true, then by Lemma 3.4 one would find a closed linear hyperplane separating c from this cone. Therefore, by Corollary 3.2, there would exist a smooth $i\bar{\partial}\partial$-closed $(1,1)$-form $\eta \in A^{1,1}(X)$ such that

1. $\langle \eta, \gamma_0 \rangle > 0$ and $\langle \eta, D \rangle > 0$ for all $D \in \mathcal{D}ou(X)^{\mathrm{irr}}_-$
2. $\langle \eta, c \rangle < 0$.

One has $([\eta] + t\gamma_0)^2 = [\eta]^2 + 2t[\eta] \cdot \gamma_0$, which becomes positive for sufficiently large t. Therefore $[\eta] + t\gamma_0$ satisfies the three assumptions of the ampleness criterion given by Theorem 2.6. This would give a Gauduchon metric g on X with

$$\langle \omega_g, c \rangle = \langle [\eta] + t\gamma_0, c \rangle = \langle [\eta], c \rangle < 0 \ ,$$

which contradicts the hypothesis. ■

Putting together Theorem 3.5 and Proposition 2.5, we get

Corollary 3.6 *Let X be a surface satisfying the assumptions of Theorem 3.5, and let u be a real, closed $(1,1)$-current. Then the following conditions are equivalent.*

1. *$\langle u, \omega_g \rangle \geq 0$ for every Gauduchon metric g on X.*
2. *The Bott-Chern cohomology class $[u] \in H^{1,1}_{BC}(X, \mathbb{R})$ of u belongs to the cone $C(\{\gamma_0\} \cup \mathcal{D}ou(X)^{\mathrm{irr}}_-)$*
3. *The Bott-Chern cohomology class $[u] \in H^{1,1}_{BC}(X, \mathbb{R})$ of u is represented by a closed positive current.*

Following the standard terminology used in the algebraic and Kählerian case [De], we define

Definition 3.7
A Bott-Chern class $c \in H^{1,1}_{BC}(X, \mathbb{R})$ will be called

1. *pseudo-effective, if it is represented by a positive current.*
2. *effective, if it decomposes as a finite linear combination* $c = \sum_{i=1}^{k} t_i[D_i]$*, where* $t_i \in \mathbb{R}_{\geq 0}$ *and* $[D_i]$ *are Bott-Chern classes associated with irreducible effective divisors* D_i*.*

Similarly, one introduces the notions of pseudo-effectiveness and effectiveness for a de Rham cohomology class $c \in H^{1,1}(X,\mathbb{R})$*.*

We denote by $\mathcal{P}(X)$, $\mathcal{E}(X) \subset H^{1,1}_{BC}(X,\mathbb{R})$ the cones of (pseudo-)effective Bott-Chern classes.

Corollary 3.8 *Let* X *be a complex surface with* $b_1(X)$ *odd. Then*

1. $\mathcal{P}(X) = C(\{\gamma_0\} \cup \mathcal{D}ou(X)^{\mathrm{irr}}_{-})$
2. *The inclusion* $\mathcal{E}(X) \subset \mathcal{P}(X)$ *is an equality if and only if* X *has an effective divisor representing the trivial real homology class (i.e. an effective divisor* D *with* $D^2 = 0$*).*
3. *If* $c \in \mathcal{P}(X)$*, then the de Rham cohomology class of* c *is effective.*

Proof: The first statement follows directly from the previous corollary. For the second, note that when X does not admit any effective divisor representing the trivial real homology class, then $\gamma_0 \notin \mathcal{E}(X)$. Indeed, suppose that γ_0 decomposes in Bott-Chern cohomology as $\gamma_0 = \sum_{i=1}^{k} t_i[D_i]$, where D_i are irreducible effective divisors and $t_i > 0$. Consider the subspaces

$$L := \left\{ (a_1, \dots a_k) \in \mathbb{R}^k | \ \sum_i a_i[D_i] = 0 \text{ in } H^2(X,\mathbb{R}) \right\} \subset \mathbb{R}^k$$

$$L_{\mathbb{Q}} := \left\{ (a_1, \dots a_k) \in \mathbb{Q}^k | \ \sum_i a_i[D_i] = 0 \text{ in } H^2(X,\mathbb{R}) \right\} \subset \mathbb{Q}^k$$

Since the cohomology classes $[D_i] \in H^2(X,\mathbb{R})$ are rational, it follows that $L = L_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{R}$ so that $L_{\mathbb{Q}} \cap \mathbb{Q}^k_{>0}$ is dense in $L \cap \mathbb{R}^k_{>0}$. Therefore, we can find positive rationals q_i such that $\sum_i q_i[D_i] = 0$ in $H^{1,1}(X,\mathbb{R})$; this obviously gives an effective divisor representing the trivial real homology class. The third statement follows from the first. ■

Remark 3.9 *There exist complex surfaces with* $b_1(X)$ *odd admitting effective divisors representing the trivial real homology class, but admitting no irreducible effective divisors with this property.*

Indeed, let X be an *exceptional compactification of an affine line bundle over an elliptic curve* [Na1]. Such a surface belongs to class VII and contains a cycle $C = \sum D_i$ of $b_2(X)$ rational curves D_i having $D_i^2 = -2$, $C^2 = 0$. Every homological trivial effective divisor of such a surface is a positive integer multiple of the cycle C.

The third statement in Corollary 3.8 above has the following important consequence, which can be regarded as a strong existence criterion for curves on non-Kählerian surfaces.

Corollary 3.10 *Let $\mathcal{L}$ be a holomorphic line bundle over a complex surface X with $b_1(X)$ odd. Suppose that $\deg_g(\mathcal{L}) \geq 0$ for every Gauduchon metric g on X. There exists $n \in \mathbb{N}^*$ such that the de Rham Chern class $nc_1^{DR}(\mathcal{L}) \in H^{1,1}(X,\mathbb{R})$ is represented by an effective divisor.*

Proof: If $\deg_g(\mathcal{L}) \geq 0$ for every Gauduchon metric g on X, then the Chern class $c_1^{BC}(\mathcal{L})$ in Bott-Chern cohomology is pseudo-effective, so it decomposes as

$$c_1^{BC}(\mathcal{L}) = t_0[\gamma_0] + \sum_{D\in \mathcal{D}ou(X)^{\mathrm{irr}}_-} t_D[D] \ .$$

with coefficients t_0, $t_D \geq 0$. Therefore, for the de Rham Chern class, one gets

$$c_1^{DR}(\mathcal{L}) = \sum_{D\in \mathcal{D}ou(X)^{\mathrm{irr}}_-} t_D[D] \ . \tag{4}$$

On the other hand $c_1^{DR}(\mathcal{L})$, $c_1^{DR}(\mathcal{O}(D))$, $D \in \mathcal{D}ou(X)^{\mathrm{irr}}_-$ belong to the $\mathbb{Q}$-vector space $H^2(X,\mathbb{Q})$. Putting $d := \#(\mathcal{D}ou(X)^{\mathrm{irr}}_-)$, we see that the set A of real systems $(t_D)_{D\in\mathcal{D}ou(X)^{\mathrm{irr}}_-}$ satisfying (4) is an affine subspace of $\mathbb{R}^d$ defined by a linear system with rational coefficients. Therefore $A\cap\mathbb{Q}^d_{\geq 0}$ is dense in $A\cap\mathbb{R}^d_{\geq 0}$, so one can find rational non-negative coefficients satisfying (4). ∎

4 Applications

4.1 The total Ricci scalar of a non-Kählerian surface

Let X be a complex surface. Let g be a Hermitian metric g on X, A_g the corresponding Chern connection on the holomorphic tangent bundle Θ_X, and s_g the Ricci scalar of g which is defined by

$$s_g := i\Lambda_g \mathrm{Tr} F_{A_g}$$

(see [G]). The total Ricci scalar of g is

$$\sigma_g := \int_X s_g vol_g = \int_X i\omega_g \wedge \mathrm{Tr} F_{A_g} \ .$$

For a Gauduchon metric g, one has the following important interpretation of the total Ricci scalar

$$\sigma_g = 2\pi \deg_g(\Theta_X) = -2\pi\deg_g(\mathcal{K}_X) \ , \tag{5}$$

where Θ_X is the holomorphic tangent bundle of X, $\mathcal{K}_X = \det(\Theta_X)^\vee$ is the canonical line bundle and $\deg_g : \mathrm{Pic}(X) \to \mathbb{R}$ is the Gauduchon degree associated with g ([G], [LT]).

The purpose of this section is to describe explicitly the set

$$\sigma(X) := \{\gamma_g |\, g \in \mathcal{G}(X),\ \int_X vol_g = 1\} = \{-2\pi \deg_g(\mathcal{K}_X) |\, g \in \mathcal{G}(X),\ \int_X \omega_g^2 = 2\}\ ,$$

and to decide whether it is a deformation invariant or not.

Lemma 4.1 *For a Bott-Chern cohomology class $u \in H^{1,1}_{BC}(X)$ on surface with odd first Betti number put*

$$\sigma(u) := \{h \cdot u |\ h \in \mathcal{H}^+(X),\ h^2 = 1\}\ .$$

Then

$$\sigma(u) = \begin{cases} 0 & \text{when} & u = 0 \\ (0,\infty) & \text{when} & u \in \mathcal{P}(X) \setminus \{0\} \\ (-\infty, 0) & \text{when} & u \in -\mathcal{P}(X) \setminus \{0\} \\ (-\infty,\infty) & \text{when} & u \notin \mathcal{P}(X) \cup (-\mathcal{P}(X))\ . \end{cases}$$

Proof: An element $u \in \mathcal{P}(X) \setminus \{0\}$ obviously satisfies $\sigma(u) \subset (0,\infty)$. For the converse inclusion we proceed as follows:

Let ω_0 be the Kähler form of a fixed Gauduchon metric g_0. Our description of the positive cone $\mathcal{H}_+$ shows that the whole half-line

$$\left\{[\omega_0] + t\gamma_0 \,\middle|\, ([\omega_0] + t\gamma_0)^2 > 0\right\} = \left\{[\omega_0] + t\gamma_0 \,\middle|\, [\omega_0]^2 + 2t\omega_0 \cdot \gamma_0 > 0\right\}$$

is contained in $\mathcal{H}_+$. Put $h_t := [\omega_0] + t\gamma_0$ for $t > -\frac{\omega_0^2}{2\omega_0 \cdot \gamma_0}$. It suffices to notice that, for any $u \in \mathcal{P}(X) \setminus \{0\}$, one has

$$\left\{ \frac{1}{\sqrt{h_t^2}}\ h_t \cdot u \ \middle|\ t > -\frac{\omega_0^2}{2\omega_0 \cdot \gamma_0} \right\} =$$

$$= \left\{ \frac{1}{\sqrt{[\omega_0]^2 + 2t\omega_0 \cdot \gamma_0}}\ \omega_0 \cdot u \ \middle|\ t > -\frac{\omega_0^2}{2\omega_0 \cdot \gamma_0} \right\} = (0,\infty)\ .$$

This proves the first three equalities.

For the fourth, suppose that $H^{1,1}_{BC}(X,\mathbb{R}) \ni u \notin \mathcal{P}(X) \cup (-\mathcal{P}(X))$. By Corollary 3.6 there exist Gauduchon metrics g_1, g_2 such that the corresponding Kähler forms satisfy

$$[\omega_1] \cdot u > 0\ ,\ [\omega_2] \cdot u < 0$$

Modifying the two classes $[\omega_1]$, $[\omega_2]$ by $t\gamma_0$ as above, one gets easily two half-lines l_1, $l_2 \subset \mathcal{H}_+$ such that $l_1 \cdot u = (0,\infty)$, $l_2 \cdot u = (-\infty, 0)$. ■

By Lemma 4.1 and formula (5), the set $\sigma(X)$ is determined by the position of the Chern class $c_1^{BC}(\mathcal{K}_X)$ in Bott-Chern cohomology with respect to the cones

$\pm\mathcal{P}(X) = \pm C(\{\gamma_0\} \cup \mathcal{D}ou(X)^{\mathrm{irr}}_{-})$.

Taking into account that algebraic surfaces of Kodaira dimension $-\infty$ allow Kähler metrics with positive total scalar curvature, the following remark might be surprising:

Remark 4.2 *Let X be any class VII surface whose minimal model is an Inoue surface. Then the class $c_1^{BC}(\mathcal{K}_X)$ is non-zero and pseudo-effective, hence $\sigma(X) = (-\infty, 0)$.*

Proof: Let $\mathbb{H}$ be the upper half-plane. An Inoue surface is a quotient of $\mathbb{H} \times \mathbb{C}$ by a properly discontinuous group G of affine transformations. There are three classes of Inoue surfaces [In], denoted by $S^{\pm}_M$, $S^{+}_{N,p,q,r,t}$, $S^{-}_{P,p,q,r}$. Here $M \in SL(3,\mathbb{Z})$ is a matrix with a single real eigenvalue $\alpha > 1$, $N \in SL(2,\mathbb{Z})$ has two positive real eigenvalues $\alpha^{-1}, \alpha > 1$, $P \in GL(2,\mathbb{Z})$ has two real eigenvalues $\alpha > 1$ and $-\alpha^{-1}$. The symbols r, t denote numbers $r \in \mathbb{Z} \setminus \{0\}$, $t \in \mathbb{C}$ whereas p, q are real numbers satisfying a certain integrality condition.

Taking into account the way in which the group acts on pairs $(w,z) \in \mathbb{H} \times \mathbb{C}$, one checks easily that in the case of the surfaces $S^{+}_{N,p,q,r,t}$, the form $\frac{dw \wedge dz}{\mathrm{Im}(w)}$ descends to a differentiable nowhere vanishing $(2,0)$-form on S. This shows that setting

$$h_{(w,z)}(dw \wedge dz) = \mathrm{Im}(w)^2$$

one gets a Hermitian metric on the line bundle $\mathcal{K}_S$. Therefore, setting $w = u + iv$, we see that the form

$$\frac{i}{\pi}\bar\partial\,\frac{\partial v}{v} = \frac{i}{\pi}\bar\partial(-\frac{i}{v}dw) = \frac{1}{\pi}\bar\partial(\frac{1}{v}dw) = \frac{i}{\pi}(-\frac{1}{v^2}d\bar w \wedge dw) = \frac{i}{\pi v^2}dw \wedge d\bar w$$

descends to a closed $(1,1)$-form representing the Chern class of $\mathcal{K}_S$ in Bott-Chern cohomology. But this form is a non-trivial positive current. This shows that $c_1^{BC}(\mathcal{K}_S)$ is non-zero and pseudo-effective. For an Inoue surface S of type $S^{-}_{P,p,q,r}$ the formula $h_{(w,z)}(dw \wedge dz) = \mathrm{Im}(w)^2$ still defines a Hermitian metric on $\mathcal{K}_S$ whose Chern form is positive. For $S^{\pm}_M$ on takes $h_{(w,z)}(dw \wedge dz) = \mathrm{Im}(w)^1$.

For a blown up Inoue surface $X \xrightarrow{p} S$, one just notices that

$$c_1^{BC}(\mathcal{K}_X) = p^*(c_1^{BC}(\mathcal{K}_S)) + [E] \ ,$$

where E is an effective divisor. ■

For a Hopf surface, one has:

Remark 4.3 *Any primary Hopf H has an anti-canonical divisor. Therefore for such a surface one has $\sigma(H) = (0, \infty)$.*

[1] I am indebted to V. Apostolov and G. Dloussky for pointing out that the surfaces $S^{\pm}_M$, $S^{-}_{P,p,q,r}$ require a slightly different argument

Proof: A primary a Hopf surface H of the form $\mathbb{C}^2 \setminus \{0\}/\langle T \rangle$, where

$$T : (z_1, z_2) \mapsto (\alpha_1 z_1, \alpha_2 z_2)$$

(where $0 < |\alpha_1| \leq |\alpha_2| < 1$) has $\mathcal{K}_H = \mathcal{O}_H(-C_1 - C_2)$, where C_i are the elliptic curves defined by the equations $z_i = 0$. If T has the form

$$(z_1, z_2) \mapsto (\alpha_1 z_1 + a z_2^m, \alpha_2 z_2)$$

where $\alpha_2^m = \alpha_1$, one has $\mathcal{K}_H = \mathcal{O}_H(-(m+1)C)$ where C is the elliptic curve defined by the equation $z_2 = 0$. ■

For a blown up Hopf surface the result is more complicated.

Proposition 4.4 *Let X be class VII surface with $b_2(X) > 0$ whose minimal model is a primary Hopf surface. Then $c_1^{BC}(\mathcal{K}_X) \notin \mathcal{P}(X) \cup (-\mathcal{P}(X))$. In particular $\sigma(X) = (-\infty, \infty)$.*

Proof: For simplicity we give the proof only for a single blow up. Let H be a primary Hopf surface with anti-canonical effective divisor A, let $\pi : X \to H$ the blow up at a point $x_0 \in H$ with exceptional divisor E, and denote by $\tilde{A}$ the proper transform of A. Then $\mathcal{K}_X$ decomposes as

$$\mathcal{K}_X = \mathcal{O}_X(-D) \otimes \mathcal{O}_X(E) \ ,$$

where $\mathcal{O}(D) = \pi^*(\mathcal{O}(A))$, $D = \tilde{A} + kE$ (where $k \geq 0$ is the incidence order between x_0 and A).

Since D is homologically trivial, we get

$$c_1^{BC}(\mathcal{K}_X) = -t_0\gamma_0 + [E] \ .$$

Since E is the only irreducible effective divisor with negative self-intersection and γ_0, $[E]$ are linearly independent in $H^{1,1}_{B,C}(X, \mathbb{R})$, we see easily that $c_1^{BC}(\mathcal{K}_X) \notin \mathcal{P}(X) \cup (-\mathcal{P}(X))$. ■

Remark: There is standard way to endow a blown up surface $\pi : \hat{X}_{x_0} \to X$ with a Gauduchon metric (see [Bu1], [LT]). The idea is to choose a Gauduchon metric g on X and to note that there exists a closed $(1,1)$-form η representing the class of the exceptional curve E whose restriction to this curve is the opposite of its Fubiny-Study volume form. It will follow that, for all sufficiently small $\varepsilon > 0$, the form $\pi^*(\omega_g) - \varepsilon\eta$ is positive and $i\bar{\partial}\partial$-closed, so it corresponds to a Gauduchon metric $\hat{g}_\varepsilon$ on $\hat{X}_{x_0}$.

The volume of the exceptional divisor with respect to a metric $\hat{g}_\varepsilon$ is small. Therefore, in this way one gets Gauduchon metrics with *positive* total Ricci scalars on blown up Hopf surfaces; it is not clear at all how to construct explicitly Gauduchon metrics with *negative* total Ricci scalars on these surfaces.

For the minimal case one has:

Proposition 4.5 *Let X be a minimal class VII surface with $b_2(X) > 0$. The class $c_1^{BC}(\mathcal{K}_X)$ cannot be pseudo-effective. Therefore, such a surface has either $\sigma(X) = (-\infty, \infty)$ (when $c_1^{BC}(\Theta_X)$ is not pseudo-effective) or $\sigma(X) = (0, \infty)$ (when $c_1^{BC}(\Theta_X)$ is pseudo-effective).*

Proof: Suppose that $c_1^{BC}(\mathcal{K}_X)$ was pseudo-effective. By Corollary 3.10, there exists $n \in \mathbb{N}^*$ such that $nc_1^{DR}(\mathcal{K}_X) = PD([E])$ for an effective divisor $E \subset X$. This gives $\langle c_1^{DR}(\mathcal{K}_X), [E] \rangle = \frac{1}{n} c_1^{DR}(\mathcal{K}_X)^2 = -\frac{1}{n} b_2(X) < 0$, which contradicts Lemma 1.1 in [Na3]. ■

There exist many minimal class VII surfaces with pseudo-effective $c_1^{BC}(\Theta_X)$, for instance the surfaces allowing a pluri-anticanonical divisor. A hyperbolic Inoue surface X [Na1], [Na2] has two cycles A, B of rational curves, and one has $\mathcal{K}_X \simeq \mathcal{O}_X(-A - B)$ (see Lemma 2.8 in [Na1] and the proof of Lemma 4.7). Similarly, a *half Inoue surface* X [Na1] has a cycle C of $b_2(X)$ rational curves and an order two flat line bundle $\mathcal{L}$ such that $\mathcal{K}_X \simeq \mathcal{L} \otimes \mathcal{O}_X(-C)$; thus $2C$ is a bi-anti-canonical divisor.

There also exist minimal class VII-surfaces X with

1. pseudo-effective Bott-Chern class Chern class $c_1^{BC}(\Theta_X)$ but allowing no pluri-anticanonical divisors,
2. non-pseudo-effective $c_1^{BC}(\Theta_X)$.

Any *known* minimal class VII surface with $b_2 > 0$ is the special fibre X_0 of a family of surfaces $\mathcal{X} \to D$ whose fibres X_t, $t \neq 0$ are all blown up primary Hopf surfaces. If the GSS conjecture was true (which has been proved for $b_2 = 1$ [Te2]), *any* minimal class VII surface with $b_2 > 0$ would be a degeneration of a family of blown up primary Hopf surfaces.

Corollary 4.6 *The assignment $X \mapsto \sigma(X) \subset \mathbb{R}$ is not a deformation invariant for class VII surfaces. More precisely there exist families $\mathcal{X} \to D$ of such surfaces such that $\sigma(X_t) = (-\infty, \infty)$ for any $t \neq 0$ and $\sigma(X_0) = (0, \infty)$.*

Proof: It suffices to consider a one parameter family of blown up primary Hopf surfaces degenerating to a minimal class VII surface with pseudo-effective $c_1^{BC}(\Theta_X)$ (for instance a hyperbolic Inoue surface).

4.2 The stability of the canonical extension of a class VII-surface

Class VII surfaces are not completely classified yet. The main obstacle is the "Global Spherical Shell (GSS) conjecture" ([Na2], p. 220) which states that any minimal class VII surface X with $b_2(X) > 0$ contains a global spherical shell, i.e. an open submanifold $S \subset X$ biholomorphic to a neighborhood of S^3 in $\mathbb{C}^2$ such that $X \setminus S$ is connected. Minimal class VII surfaces containing a global spherical shell are well understood; any such surfaces X contains $b_2(X)$ rational curves, but there are many possibilities for the intersection graph of these curves. This intersection graph is *not* a deformation invariant. By a fundamental result of Dloussky-Oeljeklaus-Toma [DOT], any minimal class VII surface X which has $b_2(X)$ rational curves does contain a GSS, so the classification of class VII surfaces reduces to the question: " does any minimal class VII surface with $b_2(X) > 0$ possess $b_2(X)$ rational curves"?

Let X be an arbitrary class VII surface. By Serre duality $h^1(\mathcal{K}_X) = 1$, so there exists a (up to isomorphy) unique rank 2-holomorphic bundle $\mathcal{A}$ which is the central term of a nontrivial extension

$$0 \longrightarrow \mathcal{K}_X \xrightarrow{s} \mathcal{A} \xrightarrow{t} \mathcal{O}_X \longrightarrow 0 \ ,$$

which will be called *the canonical extension of* X. The problem treated in this section is: does there exist Gauduchon metrics on X with respect to which $\mathcal{A}$ is stable? The problem is not easy: when $\deg_(\mathcal{K}_X) < 0$, the obvious line subbundle $\mathcal{K}_X$ of $\mathcal{A}$ does not destabilize it, but it is of course possible that $\mathcal{A}$ is destabilized by another line bundle. This would imply that $\mathcal{A}$ can be written as extension in a different way. On the other hand, we will see that *writing a rank 2-bundle as an extension in two different ways, implies the existence of effective divisors with very special properties.* Therefore, the stability of $\mathcal{A}$ is related to the existence of curves on the base manifold X. This is an important remark because, by Dloussky-Oeljeklaus-Toma's theorem, the GSS conjecture reduces to the existence of "sufficiently many" curves.

Example: Let X be an Inoue-Hirzebruch surface (a hyperbolic Inoue surface) [Na1]. Such a surface has two disjoint cycles A, B of rational curves, and $\mathcal{K}_X \simeq \mathcal{O}_X(-A-B)$ ([Na1] p. 402, 419). We state that, in this case one has $\mathcal{A} \simeq \mathcal{O}(-A) \oplus \mathcal{O}(-B)$, so the canonical extension of such a surface is non-stable with respect to any Gauduchon metric. Indeed, one has $\mathcal{K}_X^\vee \otimes [\mathcal{O}(-A) \oplus \mathcal{O}(-B)] = \mathcal{O}(B) \oplus \mathcal{O}(A)$ so, since $A \cap B = \emptyset$, one gets a bundle embedding $\mathcal{K}_X \hookrightarrow \mathcal{O}(-A) \oplus \mathcal{O}(-B)$. Therefore, $\mathcal{O}(-A) \oplus \mathcal{O}(-B)$ is an extension of $\mathcal{O}_X$ by $\mathcal{K}_X$, and this extension cannot be trivial because $H^0(\mathcal{O}(-A) \oplus \mathcal{O}(-B)) = 0$.

Let $(e_1, \dots e_{b_2(X)})$ be a basis of $H^2(X, \mathbb{Z})/\mathrm{Tors}$ such that $e_i^2 = -1$ and $c_1^{\mathbb{Q}}(\mathcal{K}_X) = \sum_i e_i$. the existence of such a basis follows easily (see [Te2]) from Donaldson's theorem on smooth manifolds with definite intersection form [Do]. For a subset $I \subset \{1, \dots, b_2(X)\}$ we put

$$e_I := \sum_{i \in I} e_i \ , \ \bar{I} := \{1, \dots, b_2(X)\} \setminus I \ .$$

Lemma 4.7 *Let $\mathcal{E}$ be any holomorphic 2-bundle with $\det(\mathcal{E}) = \mathcal{K}_X$, $c_2(\mathcal{E}) = 0$ and let $j : \mathcal{L} \hookrightarrow \mathcal{E}$ a rank 1 locally free subsheaf with torsion free quotient.*

Then j is a bundle embedding (i.e. fibrewise injective) and there exists a subset $I \subset \{1, \dots, b_2(X)\}$ such that $c_1^{\mathbb{Q}}(\mathcal{L}) = e_I$.

Proof: The inclusion $\mathcal{L} \hookrightarrow \mathcal{E}$ fits in an exact sequence

$$0 \longrightarrow \mathcal{L} \xrightarrow{j} \mathcal{E} \xrightarrow{k} \mathcal{K}_X \otimes \mathcal{L}^{-1} \otimes \mathcal{I}_Z \longrightarrow 0 ,$$

where $Z \subset X$ is a codimension 2 locally complete intersection. Decomposing $c_1^{\mathbb{Q}}(\mathcal{L}) = \sum_i a_i e_i$ (with $a_i \in \mathbb{Z}$), this gives

$$0 = c_2(\mathcal{E}) = |Z| + \sum a_i(a_i - 1) ,$$

which happens iff $Z = \emptyset$ and $a_i \in \{0, 1\}$ for all $i \in \{1, \dots, b_2(X)\}$. ■

Proposition 4.8 *Let S be an arbitrary complex surface, let*

$$0 \longrightarrow \mathcal{L} \xrightarrow{a} \mathcal{E} \xrightarrow{b} \mathcal{O}_S \longrightarrow 0 \tag{6}$$

an exact sequence, and $\varepsilon := \delta_h(1) \in H^1(\mathcal{L}) = \mathrm{Ext}^1(\mathcal{O}_X, \mathcal{L})$ the corresponding extension invariant, where δ_h stands for the connecting operator in the associated cohomology sequence. Let $D \subset X$ a (possibly empty, possible non-reduced) effective divisor, and $u : \mathcal{O}_S(-D) \to \mathcal{O}_S$ the canonical morphism. Let $V \subset \mathrm{Hom}(\mathcal{O}_S(-D), \mathcal{E}) = H^0(\mathcal{E}(D))$ be the set of liftings of u to $\mathcal{E}$. Then

1. *If non-empty, V is an affine space modeled over the vector space $H^0(\mathcal{L}(D))$.*
2. *The restriction $v|_D \in H^0(\mathcal{E}_D(D))$ of any lifting $v \in V$ to D belongs to the subspace $H^0(\mathcal{L}_D(D))$ of $H^0(\mathcal{E}_D(D))$, so it defines an element*
$$\rho(v) \in H^0(\mathcal{L}_D(D)) .$$
3. *For every $v \in V$, the element $\rho(v)$ is a lifting of ε via the connecting operator*
$$\delta_v : H^0(\mathcal{L}_D(D)) \longrightarrow H^1(\mathcal{L})$$
associated with the exact sequence $0 \to \mathcal{L} \to \mathcal{L}(D) \to \mathcal{L}_D(D) \to 0$.
4. *The map $V \ni v \mapsto \rho(v) \in H^0(\mathcal{L}_D(D))$ defines a bijection between the quotient $V/H^0(\mathcal{L})$ and the space H_ε of δ_v-liftings of ε in $H^0(\mathcal{L}_D(D))$. Here $H^0(\mathcal{L})$ was regarded as a subspace of the model vector space $H^0(\mathcal{L}(D))$.*
5. *The vanishing locus $Z(v) \subset S$ of v is contained in D and coincides with the vanishing locus $Z(\rho(v))) \subset D$. In particular, the lifting $v :_S \mathcal{O}(-D) \to \mathcal{E}$ of u is a bundle embedding if and only if $\rho(v)$ is a trivialization of the line bundle $\mathcal{L}_D(D)$ over D.*

Proof: The first statement is obvious. For the second, use the following sheaf diagram with exact rows and exact columns

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{L} & \xrightarrow{a} & \mathcal{E} & \xrightarrow{b} & \mathcal{O}_S & \longrightarrow & 0 \\
 & & \downarrow i & & \downarrow i' & & \downarrow i'' & & \\
0 & \longrightarrow & \mathcal{L}(D) & \xrightarrow{a'} & \mathcal{E}(D) & \xrightarrow{b'} & \mathcal{O}_S(D) & \longrightarrow & 0 \\
 & & \downarrow p & & \downarrow p' & & \downarrow p'' & & \\
0 & \longrightarrow & \mathcal{L}_D(D) & \xrightarrow{a''} & \mathcal{E}_D(D) & \xrightarrow{b''} & \mathcal{O}_D(D) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array} \qquad (7)$$

and note that the image of $u \in H^0(\mathcal{O}_S(D))$ in $H^0(\mathcal{O}_D(D))$ vanishes, so the image of v in $\mathcal{E}_D(D)$ belongs to $\mathcal{L}_D(D)$.

3. The first row and the first column in the diagram (7) can be regarded as resolutions of the rank 1 locally free sheaf $\mathcal{L}$. This diagram also yields a third resolution of the same sheaf, namely the simple (or total) complex associated with the double complex (7).

$$0 \to \mathcal{L} \xrightarrow{(a,i)} \mathcal{E} \oplus \mathcal{L}(D) \xrightarrow{A} \mathcal{O}_S \oplus \mathcal{E}(D) \oplus \mathcal{L}_D(D) \xrightarrow{B} \mathcal{O}_S(D) \oplus \mathcal{E}_D(D) \xrightarrow{C} \mathcal{O}_D(D) \to 0 .$$

Truncating this resolution, one gets the short exact sequence

$$0 \to \mathcal{L} \xrightarrow{(a,i)} \mathcal{E} \oplus \mathcal{L}(D) \xrightarrow{A} \mathrm{im}(A) = \ker(B) \to 0 . \qquad (8)$$

The idea of the proof is to notice that a lift $v \in V$ of u defines an element $r(v)$ in $H^0(\mathrm{im}(B))$, namely

$$r(v) = (1, v, \rho(v)) .$$

Let ∂ be the connecting operator associated with the short exact sequence (8). One can compute $\partial(r(v)) \in H^1(\mathcal{L})$ in two ways: comparing the exact sequence (8) with the first row in (7) and using the functoriality of the connecting operator, one gets $\partial(r(v)) = \delta_h(1) = \varepsilon$, whereas comparing (8) with the first column in (7), one has $\partial(r(v)) = \delta_v(v)$.

4. If $\rho(v) = \rho(v')$, the $v - v' \in H^0(\mathcal{E}_D(D))$ is mapped to 0 via both vertical an horizontal arrows in (7). A simple diagram chasing shows that $v - v'$

comes from $H^0(\mathcal{L})$ via the obvious morphism. This proves the injectivity. Let now $w \in H^0(\mathcal{L}_D(D))$ be an element which is mapped to ε via δ_v.

For surjectivity, let $w \in H^0(\mathcal{L}_D(D))$ be a lift of ε via δ. Since $\varepsilon = \delta_v(w)$, it follows that the image of ε in $H^1(\mathcal{L}(D))$ vanishes. Similarly, since $\varepsilon = \delta_h(1)$, the image of ε in $H^1(\mathcal{E})$ will vanish, too. Therefore, in the cohomology sequence associated with (8), ε is mapped to 0 in $\mathcal{E} \oplus \mathcal{L}(D)$, so ε can be written as $\partial(\theta)$, for an element $\theta = (\varphi, \psi, \chi) \in H^0(\ker(B))$. Using again the functoriality of the connecting operator, we see that $\delta_h(\varphi) = \delta_v(\chi) = \varepsilon = \delta_h(1) = \delta_v(w)$. We can modify the triple θ by a suitable element in $A(H^0(\mathcal{E}) \oplus H^0(\mathcal{L}(D)))$ to get a lift θ' of ε via ∂, having the first component 1 and the third component w. The second component v of θ' will satisfy $\rho(v) = w$.

5. Consider, in general, an epimorphism $\pi : \mathcal{F} \to \mathcal{G}$ of holomorphic vector bundles over a complex space Y, and let σ be a holomorphic section of $\mathcal{F}$. the vanishing locus $Z(\sigma)$ of σ is the complex subspace of Y defined by the ideal sheaf $\sigma^\vee(\mathcal{F}^\vee) \subset \mathcal{O}_Y$. One has the following useful associativity property:

$$Z(\sigma) = Z(\sigma|_{Z(\pi\circ\sigma)}) \ ,$$

where the restriction $\sigma|_{Z(\pi\circ\sigma)}$ can be regarded as a section in the holomorphic bundle $\ker(\pi)|_{Z(\pi\circ\sigma)}$.

The result follows by applying this associativity principle to the epimorphism $b' : \mathcal{E}(D) \to \mathcal{O}_S(D)$ and to notice that $Z(b' \circ v) = Z(u) = D$. For the second statement, note that $\mathcal{L}_D(D)$ is a line bundle on D, so the vanishing locus of a section in $\mathcal{L}_D(D)$ is empty if and only if defines a global trivialization of this line bundle (the condition that it does not vanish at any point is not sufficient for a non-reduced divisor D). ■

Corollary 4.9 *With the notations and in the conditions of Proposition 4.8, the natural map $\mathcal{O}(-D) \to \mathcal{O}_X$ can be lifted to a bundle embedding $\mathcal{O}(-D) \hookrightarrow \mathcal{E}$ if and only if there exists a section $\alpha \in H^0(\mathcal{L} \otimes \mathcal{O}_D(D))$ with the properties*

1. *α defines a trivialization of $\mathcal{L} \otimes \mathcal{O}_D(D)$,*
2. *The image of α in $H^1(\mathcal{L})$ via the connecting operator δ_v is the invariant ε of the given extension (6).*

Corollary 4.10 *Let X be a minimal class VII surface with $b_2(X) > 0$ and $\mathcal{A}$ its canonical extension. The bundle $\mathcal{A}$ can be written as an extension*

$$0 \longrightarrow \mathcal{M} \longrightarrow \mathcal{A} \longrightarrow \mathcal{K}_X \otimes \mathcal{M}^{-1} \longrightarrow 0$$

iff and only if there exists a non-empty effective divisor $D \subset X$ satisfying the following properties:

1. *$\mathcal{M} \simeq \mathcal{O}_X(-D)$,*

2. $\mathcal{K}_X \otimes \mathcal{O}_D(D) \simeq \mathcal{O}_D$.

3. $h^0(\mathcal{K}_X \otimes \mathcal{O}_D(D)) - h^0(\mathcal{K}_X \otimes \mathcal{O}(D)) = 1$

Proof:

If $\mathcal{M}$ is a line subbundle of $\mathcal{A}$, it must admit a non-trivial map to $\mathcal{O}_X$, so it must be isomorphic to $\mathcal{O}(-D)$ for an effective divisor $D \subset X$. D must be non-empty, because the extension defining $\mathcal{A}$ does not split. The second condition is necessary by Corollary 4.10; we show that the third condition is also necessary. In our case, the cohomology exact sequence associated with the first vertical column in (7) reads

$$H^0(\mathcal{K}_X) = 0 \longrightarrow H^0(\mathcal{K}_X \otimes \mathcal{O}_X(D)) \longrightarrow H^0(\mathcal{K}_X \otimes \mathcal{O}_D(D)) \xrightarrow{\delta_v} H^1(\mathcal{K}_X) \simeq \mathbb{C}$$

On the other hand, by Corollary 4.10, the map δ_v must be non-trivial, so one has $h^0(\mathcal{K}_X \otimes \mathcal{O}_D(D)) - h^0(\mathcal{K}_X \otimes \mathcal{O}(D)) = 1$.

In order to prove that the three conditions are also sufficient, it suffices to show that they imply the existence of a trivialization of $\mathcal{K}_X \otimes \mathcal{O}_D(D)$ which is mapped onto a non-trivial element of the line $H^1(\mathcal{K}_X)$.

Since $a(X) = 0$ ([Na3] p. 477), one has $h^0(\mathcal{K}_X \otimes \mathcal{O}(D)) \leq 1$ so, by the third condition one has either $h^0(\mathcal{K}_X \otimes \mathcal{O}_D(D)) = 1$ and $h^0(\mathcal{K}_X \otimes \mathcal{O}(D)) = 0$ or $h^0(\mathcal{K}_X \otimes \mathcal{O}_D(D)) = 2$ and $h^0(\mathcal{K}_X \otimes \mathcal{O}(D)) = 1$. Taking into account that $\mathcal{K}_X \otimes \mathcal{O}_D(D) \simeq \mathcal{O}_D$, the claim is obvious in the first case because, in this case, *any* non-trivial section of $\mathcal{K}_X \otimes \mathcal{O}_D(D)$ will be a trivialization which is mapped onto a non-trivial element of $H^1(\mathcal{K}_X)$.

In the case $h^0(\mathcal{K}_X \otimes \mathcal{O}_D(D)) = 2$, $h^0(\mathcal{K}_X \otimes \mathcal{O}(D)) = 1$, it suffices to notice that both $\ker(\delta_v)$ and the subset F of $H^0(\mathcal{K}_X \otimes \mathcal{O}_D(D))$ consisting of sections which are not trivializations are proper Zarisky closed subsets of the vector space $H^0(\mathcal{K}_X \otimes \mathcal{O}_D(D))$, so the complement of their union is non-empty.

■

Let X be a minimal class VII surface. We agree to call *a cycle* of X any reduced divisor C which is either an elliptic curve, or a singular rational surface with a node, or a cycle of smooth rational curves. In the last two cases C will be called a cycle of rational curves. In all three cases the canonical sheaf $\omega_C := \mathcal{K}_X \otimes \mathcal{O}_C(C)$ is trivial.

Proposition 4.11 *Let X be a minimal class VII surface with $b_2(X) > 0$ and let $D \subset X$ be a non-empty effective divisor of X satisfying*

1. *The canonical line bundle $\omega_D := \mathcal{K}_X \otimes \mathcal{O}_D(D)$ is trivial.*

2. *$h^0(\mathcal{K}_X \otimes \mathcal{O}_D(D)) - h^0(\mathcal{K}_X \otimes \mathcal{O}(D)) = 1$.*

Then there exists $I \subset \{1, \ldots, b_2(X)\}$ such that $c_1^{\mathbb{Q}}(\mathcal{O}(-D)) = e_I$ and one of the following holds

1. *D is a cycle,*

2. $\mathcal{O}(-D) = \mathcal{K}_X$ (i.e. D is an anti-canonical divisor).

Proof: The existence of $I \subset \{1, \ldots, b_2(X)\}$ such that $c_1^{\mathbb{Q}}(\mathcal{O}(-D)) = e_I$ follows from Lemma 4.7 and Corollary 4.10.

Since $h^0(\omega_D) = h^1(\mathcal{O}_D) \geq 1$, one gets $h^1(D_{\mathrm{red}}) \geq 1$ by Lemma 2.7 in [Na1]. Let $0 < C \leq D_{\mathrm{red}}$ be minimal with the property $h^1(C) \geq 1$. By Lemma 2.3, and Lemma 2.12 in [Na1] C is either a cycle or a union of two disjoint cycles. Decompose D as $D = C + E$ for an effective divisor E. Put $\mathcal{M} := \mathcal{K}_X \otimes \mathcal{O}(D)$. Noting that $h^2(\mathcal{M}) = h^0(\mathcal{O}(-D)) = 0$, we get the exact sequence

$$0 \to H^0(\mathcal{M}(-C)) \to H^0(\mathcal{M}) \to H^0(\mathcal{M}_C) \to$$
$$\to H^1(\mathcal{M}(-C)) \to H^1(\mathcal{M}) \to H^1(\mathcal{M}_C) \to H^2(\mathcal{M}(-C)) \to 0$$

Case 1. $h^0(\mathcal{K}_X \otimes \mathcal{O}_D(D)) = 1$ and $h^0(\mathcal{K}_X \otimes \mathcal{O}(D)) = 0$.

In this case we get by Riemann-Roch theorem $h^1(\mathcal{M}) = 0$, hence (recalling that $\mathcal{M}$ is trivial on D, hence also on C)

$$h^2(\mathcal{M}(-C)) = h^1(\mathcal{M}_C) = h^1(\mathcal{O}_C) \geq 1 \ .$$

Therefore $h^0(\mathcal{O}(-E)) \geq 1$, which shows that E is empty, so $D = C$. In this case $D = C$ must be a single cycle, because otherwise one would have $h^0(\mathcal{K}_X \otimes \mathcal{O}_D(D)) = 2$.

Case 2. $h^0(\mathcal{K}_X \otimes \mathcal{O}_D(D)) = 2$ and $h^0(\mathcal{K}_X \otimes \mathcal{O}(D)) = 1$.

Note first that $c_1^{\mathbb{Q}}(\mathcal{K}_X \otimes \mathcal{O}(D)) = e_{\bar{I}}$ and $h^0(\mathcal{K}_X \otimes \mathcal{O}(D)) > 0$. Using Lemma 2.3 in [Na3] (which holds for any minimal class VII surface with positive b_2) one gets $\bar{I} = \emptyset$, so

$$\mathcal{K}_X \otimes \mathcal{O}(D) = \mathcal{O}(F) \tag{9}$$

for an effective divisor F with $F \cdot F = 0$. Therefore, D must be a *numerically* anti-canonical divisor. If X contains no homologically trivial effective divisors, then F must be empty, so D is anti-canonical as claimed.

Suppose now that X does contain homologically trivial divisors. Minimal class VII-surfaces with $b_2 > 0$ containing homologically trivial effective divisors are classified. Any such surface is an exceptional compactification of an affine line bundle over an elliptic curve [Na1], [Na2], [Na3], contains a GSS, and contains a homologically trivial cycle C of $b_2(X)$ rational curves D_i. An exceptional compactification of a non-linear affine line bundle does not contain any other curve but the irreducible components D_i of C, which all satisfy $\langle c_1(\mathcal{K}_X), D_i \rangle = 0$. Therefore, on such a surface there exist no anti-canonical numerically divisor.

The exceptional compactifications of linear line bundles are called parabolic Inoue surfaces. Such a surface contains a smooth elliptic curve Z with $Z \cdot Z = -b_2(X)$, $Z \cap C = \emptyset$. In this case one has $\mathcal{K}_X = \mathcal{O}_K(-C - Z)$ and the only homologically trivial effective divisors are nC, $n \in \mathbb{N}$. Therefore (9) would imply a linear equivalence of the form $D \sim (n+1)C + Z$. Since $a(X) = 0$, one has $D = (n+1)C + Z$.

We claim that only for $n = 0$ one can have $\omega_D = \mathcal{O}_D$, which will complete the proof. Indeed, if $\omega_{(n+1)C} = \mathcal{O}_{(n+1)C}$ then, taking into account $\omega_C = \mathcal{O}_C$, one would have $\mathcal{O}_C(nC) = \mathcal{O}_C$.

But $\mathcal{O}(C)$ is a flat line bundle on X which is associated with a representation $\rho : \mathbb{Z} \simeq \pi_1(X) \to \mathbb{C}^*$ with $|\rho(1)| \neq 1$ (see ([D] section 1.2)[2]. On the other hand, the natural map $H_1(C, \mathbb{Z}) \to H_1(X, \mathbb{Z})$ is an isomorphism ([Na1] p. 404). Therefore, for $n \geq 1$, the restriction of $\mathcal{O}(nC)$ to C is a flat line bundle associated with a nontrivial representation $\mathbb{Z} \simeq \pi_1(C) \to \mathbb{C}^*$, so $H^0(\mathcal{O}_C(nC)) = 0$. ■

Therefore, a line subbundle of $\mathcal{A}$ is either isomorphic to $\mathcal{K}_X$ or to a line bundle of the form $\mathcal{O}(-C)$ for a cycle $C \subset X$. We can prove now:

Theorem 4.12 *Let X be a minimal class VII surface with $b_2 > 0$. Suppose that $\mathcal{A}$ is unstable for any Gauduchon metric on X. Then one of the following holds:*

1. *X contains two cycles, i.e. X is either a hyperbolic or a parabolic Inoue surface [Na1].*

   In this case $\mathcal{A}$ is a direct sum of line bundles.

2. *$c_1^{BC}(\Theta_X)$ is pseudo-effective, X contains a single cycle C and the Bott-Chern class $c_1^{BC}(\mathcal{K}_X^\vee(-2C))$ is pseudo-effective.*

Proof: Let X be a minimal class VII surface with $b_2(X) > 0$ which does not contain two cycles. In other words, X is neither a hyperbolic nor a parabolic Inoue surface. We will prove that, if $\mathcal{A}$ is unstable with respect to any Gauduchon metric on X then 2. holds. By Proposition 4.5 we have to consider only the following two cases:

1. Neither $c_1^{BC}(\mathcal{K}_X)$ nor $-c_1^{BC}(\mathcal{K}_X)$ is pseudo-effective.

   It is easy to see that in this case, there do exist Gauduchon metrics g for which $\mathcal{A}$ is stable. Indeed, by Lemma 4.1, there exist Gauduchon metrics g_-, g_0 on X such that $\deg_{g_-}(\mathcal{K}_X) < 0$, $\deg_{g_0}(\mathcal{K}_X) = 0$. If X did not contain any cycle at all, then by Proposition 4.11, any line subbundle of $\mathcal{A}$ is isomorphic to $\mathcal{K}_X$, so stability is guaranteed as soon as $\deg_g(\mathcal{K}_X) < 0$. Therefore, $\mathcal{A}$ will be g_- -stable in this case.

   When X contains a single cycle C, denote $\nu := \deg_{g_0}(\mathcal{O}(C)) > 0$ and let η a closed $(1,1)$-form representing $c_1^{DR}(\mathcal{K}_X)$. For any sufficiently small $|t|$

[2] I am indebted to Georges Dloussky, who kindly explained me this important property of the line bundle $\mathcal{O}_X(C)$.

the form $\omega_t := \omega_{g_0} + t\eta$ is the Kähler metric of a Gauduchon metric g_t with $\deg_{g_t}(\mathcal{K}_X) = -tb_2(X)$, $\deg_{g_t}(\mathcal{O}(-C)) = -\nu + t\langle c_1^{DR}(\mathcal{K}_X), [C]\rangle$. For sufficiently small positive t one has

$$\deg_{g_t}(\mathcal{O}(-C)) < \deg_{g_t}(\mathcal{K}_X) < \frac{\deg_{g_t}(\mathcal{K}_X)}{2} = \mu_{g_t}(\mathcal{A}) < 0 \ ,$$

proving that neither $\mathcal{K}_X$ nor $\mathcal{O}(-C)$ destabilizes the bundle $\mathcal{A}$.

2. $-c_1^{BC}(\mathcal{K}_X)$ is pseudo-effective.

   In this case for *every* Gauduchon metric g on X the subbundles of $\mathcal{A}$ which are isomorphic to $\mathcal{K}_X$ do not g-destabilize $\mathcal{A}$.

   Therefore if $\mathcal{A}$ is unstable for every $g \in \mathcal{G}(X)$, X must contain a cycle $C \subset X$ such that

$$\deg_g(\mathcal{O}(-C)) \geq \frac{1}{2}\deg_g(\mathcal{K}_X) \ \forall g \in \mathcal{G}(X) \ .$$

By Corollary 3.6 this implies that the Bott-Chern class $c_1^{BC}(\mathcal{K}_X^\vee(2C))$ is pseudo-effective. ■

Corollary 4.13 *If $\mathcal{A}$ is unstable for any Gauduchon metric on X, then X contains a GSS.*

Proof: When X contains two cycles, it must be either a hyperbolic or a parabolic Inoue surface [Na1], so it contains a GSS. When X does not contain two cycles, $c_1^{BC}(\Theta_X)$ must be pseudo-effective. By Corollary 3.10 it follows that a multiple of the de Rham class $-c_1^{DR}(\mathcal{K}_X)$ is represented by an effective divisor. In other words, X contains a numerically pluri-anticanonical divisor. But, by the main result of [D], such a surface contains a GSS. ■

Corollary 4.13 can be reformulated as follows:

Remark 4.14 *If X was a counter-example to the GSS conjecture, X must admit Gauduchon metrics with respect to which the bundle $\mathcal{A}$ is stable.*

The surfaces X with the property "$\mathcal{A}$ is unstable for every Gauduchon metric" are very special. Indeed, either X is a (hyperbolic or parabolic) Inoue surface, or

Remark 4.15 *If X is in case 2. of Theorem 4.12 it must contain a single cycle C, and, writing $c_1^{DR}(\mathcal{O}(C)) = -e_I$ with $I \subset \{1, \dots, b_2(X)\}$, a multiple of the cohomology class $e_I - e_{\bar{I}}$ must be represented by an effective divisor.*

This follows again by Corollary 3.10 using the pseudo-effectiveness of the class $c_1^{BC}(\mathcal{K}_X^\vee(-2C))$.

Acknowledgments: I am indebted to Georges Dloussky and Nicholas Buchdahl for important suggestions and useful discussions on the subject. Many statements in the present article are direct consequences of their deep results.

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