

# COHOMOTOPY INVARIANTS AND THE UNIVERSAL COHOMOTOPY INVARIANT JUMP FORMULA

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## 1. INTRODUCTION

**1.1. Motivation.** The goal of this article is to develop a general formalism for the construction of cohomotopy invariants associated with a certain class of non-linear maps between Hilbert bundles. The main example we have in mind is the Seiberg-Witten map, but the formalism applies to other interesting classes of maps related to gauge theoretical problems as well.

The idea to define cohomotopy-type Seiberg-Witten invariants is due to Furuta, who introduced a class of refined Seiberg-Witten invariants (called “stable homotopy version of the Seiberg-Witten invariants”) in a geometric, non-formalized way [Fu2]. According to Furuta, the new invariants belong to a certain inductive limit of sets of homotopy classes of maps associated with “finite dimensional approximations” of the Seiberg-Witten map. The structure and the functorial properties of this inductive limit (with respect to diffeomorphisms between 4-manifolds) have not been worked out in this article. A precise version of the new invariants has been introduced later by Bauer-Furuta in [BF]: the Bauer-Furuta classes belong to certain stable cohomotopy groups associated with a presentation  $(E, F)$  of the K-theory element  $\text{ind}(\mathcal{D})$  defined by a fixed  $Spin^c$ -structure. This element  $\text{ind}(\mathcal{D})$  belongs to the K-theory group  $K(B)$ , where  $B = H^1(X, \mathbb{R})/H^1(X, \mathbb{Z})$  is the Picard group of the base manifold  $X$ .

In this article we propose a different construction of cohomotopy invariants which has the following advantages: Our construction yields a larger class of invariants, which in general are finer than the Bauer-Furuta classes, and have better functorial properties with respect to diffeomorphisms. In order to explain the advantages of the new formalism in a non-technical way, we begin with the following natural questions:

It is well known that the Seiberg-Witten map  $\mu$  can be regarded in a natural way as an  $S^1$ -equivariant bundle map between Hilbert bundles over the torus  $B$  (see section 3.4 in this article). We choose first the perturbing form in the second Seiberg-Witten equation in the “bad way”, i.e. such that equations have reducible solutions (solutions with trivial spinor component); we make this “bad choice” even in the case  $b_+(X) > 1$ ! In “classical Seiberg-Witten” theory one perturbs the second Seiberg-Witten equation using a nontrivial self-dual harmonic form  $\kappa \in i\mathbb{H}^+ \setminus \{0\}$ , and gets a new map  $\mu_\kappa$  which defines a moduli space which does not contain reductions. Instead of a constant perturbation  $\kappa$ , we consider a map  $\kappa : B \rightarrow i\mathbb{H}^+ \setminus \{0\}$ , and perturb the Seiberg-Witten map  $\mu$  (regarded as bundle map over  $B$ ) using this map. The associated invariant will depend on the homotopy class  $[\kappa] \in [B, S(i\mathbb{H}^+)]$ . Our questions are:

- (1) Does one obtain new invariants in this way?
- (2) If so, does one have a *universal cohomotopy invariant jump formula*, i.e. a formula which describes the jump of the cohomotopy invariant when one passes from one homotopy class to another?
- (3) Use again constant perturbation forms  $\kappa$ , but let  $\kappa$  vary in the sphere  $S(i\mathbb{H}^+)$  and regard the obtained map  $\tilde{\mu}$  as an  $S^1$ -equivariant bundle map over the larger basis  $B \times S(i\mathbb{H}^+)$ . Does this universal perturbation  $\tilde{\mu}$  yield more differential topological information than the individual perturbations  $\mu_\kappa$ ? If not, express the cohomotopy invariant associated with  $\tilde{\mu}$  in terms of the invariant associated with  $\mu_\kappa$  and topological invariants of  $X$ .

These questions are interesting as soon as  $b_1 \geq b_+ - 1$  (even for  $b_+ > 1$ !) and they are also interesting *for the classical invariant*, because for non-constant perturbations  $\kappa$  one gets new Seiberg-Witten type moduli spaces. The universal wall-crossing formula for the *full Seiberg-Witten invariant*<sup>1</sup> [OT] should be a formal consequence

<sup>1</sup>The full Seiberg-Witten invariant is an element in  $\Lambda^*(H^1(X, \mathbb{Z}))$  [OT]. The numerical Seiberg-Witten invariant (the original invariant introduced by Witten) is the degree 0 term of the full invariant.

of the universal cohomotopy invariant jump formula. These questions will be completely answered in this article. Another motivation for proposing a new formalism was the need to have invariants with clear functorial properties with respect to diffeomorphisms of 4-manifolds. The point here is that the definition of the stable cohomotopy group used in [BF] depends on the choice of a presentation  $(E, B \times \mathbb{C}^n)$  of  $\text{ind}(\mathcal{D}) \in K(P)$  (see [BF] p. 8-9). Since in general such a presentation has homotopically non-trivial automorphisms, the obtained cohomotopy groups cannot be regarded as invariants of the K-theory element  $\text{ind}(\mathcal{D}) \in K(B)$ . This makes it very difficult to control the functorial properties of the Bauer-Furuta stable cohomotopy groups with respect to homeomorphisms or diffeomorphisms of 4-manifolds. Using Segal cocycles instead of finite rank presentations ([BF] p. 7-8) does not remove the problem, because of monodromy phenomena in the space of Segal cocycles. A similar difficulty concerns the concept ‘‘Thom spectrum of a virtual bundle’’, used by Bauer-Furuta (see [BF] p. 8) and other authors in order to give a geometric interpretation of the Bauer-Furuta classes. One can indeed associate a Thom spectrum to a *fixed presentation*  $(E, B \times \mathbb{C}^n)$  of a K-theory element  $x \in K(B)$ , but unfortunately *not to  $x$  itself*. For 4-manifolds with  $b_1 = 0$ , the Bauer-Furuta class gives a well defined invariant, which can easily be identified with the image of our invariant under a boundary morphism of cohomotopy groups. The two invariants are equivalent when  $b_1 = 0$ ,  $b_+ > 1$ .

Our new point of view has the following advantages:

- (1) The new cohomotopy Seiberg-Witten invariants are finer than the full classical Seiberg-Witten invariants *in all cases*, including the case of manifolds with  $b_1 \geq b_+ - 1$  and including the invariants associated with non-constant perturbations  $\kappa : B \rightarrow i\mathbb{H}_+^2 \setminus \{0\}$ . In the case  $b_1 \geq b_+ - 1$  we prove a universal cohomotopy invariant jump formula; the universal wall-crossing formula for the classical invariant is a formal consequence of this result.
- (2) Our invariant belongs to a cohomotopy group which is intrinsically associated with the base 4-manifold (and a set of topological data), and has clear functorial properties with respect to homeomorphisms of 4-manifolds.

**1.2. Summary of results.** In the first section we construct a graded cohomotopy group associated with a K-theory element  $x \in K(B)$ . To every representative  $(E, F) \in x$  we associate the graded group  ${}_{S^1}\alpha_B^*(S(E)_{+B}, F_B^+)$ , where  ${}_{S^1}\alpha_B^*(\cdot, \cdot)$  stands for the  $S^1$ -equivariant stable cohomotopy group functor on the category of pointed  $S^1$ -spaces over  $B$ , obtained by stabilizing with spaces of the form  $(\eta \oplus \xi_0)_B^+$ , where  $\eta$  is a complex, and  $\xi_0$  a real bundle. Note that we do not use all characters of  $S^1$  in the stabilizing process; for this reason we do not use the standard notation  ${}_{S^1}\omega_B^*$  found in the literature [CJ]. We define  $\alpha^*(x)$  to be the inductive limit of the graded groups  $\alpha^*(E, F)$  with respect to the category  $\mathcal{T}(x)$  of representatives  $(E, F)$  of  $x$ . Since  $\mathcal{T}(x)$  is not a small filtered category (see [AM], section 5.1 in this article), this limit cannot be obtained using the classical construction. It will be constructed in two steps: First we stabilize the graded group  ${}_{S^1}\alpha_B^*(S(E)_{+B}, F_B^+)$  with respect to standard representative enlargements  $(E, F) \mapsto (E \oplus U, F \oplus U)$ , we obtain a new graded group  $\hat{\alpha}^*(E, F)$ , which still depends on the fixed pair  $(E, F)$ . The groups  $\hat{\alpha}^*(E, F)$ ,  $\hat{\alpha}^*(E', F')$  defined by two representatives  $(E, F)$ ,  $(E', F')$  of  $x$  are *non-canonically* isomorphic. The group  $\alpha^*(x)$  will be the quotient of  $\hat{\alpha}^*(E, F)$  by the equivalence relation generated by the inductive limit of the automorphism groups  $\text{Aut}(E \oplus U) \times \text{Aut}(F \oplus U)$ . We give an explicit description of  $\alpha^*(x)$  as a quotient

of the group  $\hat{\alpha}^*(E, F)$  associated with any representative  $(E, F)$  by the action of the image of the  $J$ -homomorphism  $S^1 J : K^{-1}(B) \rightarrow S^1 \alpha^0(B)^\times$  in the group of units  $S^1 \alpha^0(B)^\times$  of the ground ring  $S^1 \alpha^0(B) := S^1 \alpha_B^0(B_{+B}, B_{+B})$ . In other words, we are able to control the effect of bundle automorphisms in our inductive limit and we obtain a graded group which is intrinsically associated with the K-theory element  $x$  and is a sufficiently fine invariant. We believe that this construction is of independent interest from the point of view of homotopy theory.

A way to understand the role of the graded group  $\alpha^*(x)$  is the following: Because the presence of homotopically non-trivial bundle automorphisms, one cannot define the projectivization of a K-theory element  $x \in K(B)$  (neither in the category of topological spaces nor in the category of spectra). *The graded group  $\alpha^*(x)$  plays the role of what should be the cohomotopy group of a formal projectivization of the K-theory element  $x$ .*

In the second section we first introduce a distinguished class of non-linear maps  $\mu$  between Hilbert bundles over a compact base  $B$ . The  $\mathbb{C}$ -linear part of the linearization of such a map  $\mu$  at the zero section is a linear Fredholm operator, so it defines a K-theory element  $x \in K(B)$ . The goal of the section is the construction of an invariant  $[\mu] \in \alpha^*(x)$ . This invariant is constructed using *finite dimensional approximations* of the map  $\mu$ . In order to get these approximations we make also use of the retractions  $\rho_A : \mathcal{A}^+ \setminus S(\mathcal{A}^\perp) \rightarrow \mathcal{A}^+$  associated with finite dimensional subspaces  $A$  of a Hilbert space  $\mathcal{A}$ , as in [BF]. This method to construct finite dimensional approximations applies to a very large class of non-linear maps, whereas Furuta's original method based on  $L^2$ -orthogonal projections on direct sums of eigenspaces (see [Fu2]) is limited to maps whose linearizations are elliptic differential operators.

The main difference between our definition and the construction of the Bauer-Furuta classes given in [BF], is that

- (1) our construction uses only spaces fibered over the base  $B$ . In particular we avoid using Thom spaces,
- (2) we treat the real and the complex summands in our finite dimensional approximations separately.

Therefore, from this point of view, our construction is *closer to the original ideas of Furuta* [Fu2]. Having the finite dimensional approximation, a representative of the invariant is an element in a group of the form  $S^1 \alpha_B^*(S(E)_{+B}, F_B^+)$  obtained by a simple geometric construction, which we call *the cylinder construction*. Since we contract a smaller subspace, the obtained class will carry more information than the one defined in [BF]. In the same section we show that the Seiberg-Witten map associated with a  $\text{Spin}^c$ -structure  $\tau$  on a Riemannian 4-manifold  $M$  with  $b_+(M) > 0$  yields a non-linear Fredholm map  $sw_\kappa$  (depending on a twisting map  $\kappa : B = H^1(X, \mathbb{R})/H^1(X, \mathbb{Z}) \rightarrow i\mathbb{H}_+^2 \setminus \{0\}$ ) which belongs to our distinguished class of maps. Hence the general theory applies and yields a cohomotopy Seiberg-Witten invariant  $\{sw_\kappa\} \in \alpha^{b_+(M)-1}(\text{ind}(\not{D}))$ , which only depends of the homotopy class of  $\kappa$ . Our construction of the bundle map  $sw_\kappa$  is different from the one given in [BF].

Even though differs in several important points from the formalisms given in [Fu2], [BF] and [B2], and we have made use of several important tools which have been developed by these authors.

In the third section we prove several fundamental properties of the invariant  $[\mu] \in \alpha^*(x)$  in our general, abstract framework:

- (1) We study the image of our invariant under the Hurewicz morphism, and we prove that the Poincaré dual of this image can be identified with the virtual fundamental class of the vanishing locus. In other words, the homology invariant associated with the virtual fundamental class of the “moduli space” (i.e. the  $S^1$ -quotient of the vanishing locus of  $\mu$ ) can be identified with the Hurewicz image of the cohomotopy invariant.
- (2) We prove a formal universal cohomotopy invariant jump formula.
- (3) We obtain a multiplicative property of the invariant with respect to cartesian product of maps. This multiplicative property has an important corollary: the cohomotopy invariant associated with two maps belonging to our distinguished class of non-linear maps is always a 2-torsion element.

Specialized to the Seiberg-Witten map, these properties automatically yield important results for the new cohomotopy Seiberg-Witten invariants. The first result shows that the cohomotopy Seiberg-Witten invariant is refinement of the classical *full* Seiberg-Witten invariants in all cases. Combined with the second property, this also yields a universal invariant jump formula for the full classical Seiberg-Witten invariant in the case  $b_1(X) \geq b_+(X) - 1$ . Our multiplicative property gives a formula for the cohomotopy invariant of a connected sum of two 4-manifolds, even in the case when one term of the sum has  $b_+ = 0$ . The 2-torsion property of the invariant implies that the cohomotopy invariant of a connected sum of 4-manifolds  $X_i$  having  $b_+(X_i) > 0$  ( $i = 1, 2$ ) is always a 2-torsion element. We will come back to these consequences in Seiberg-Witten theory in a future article.

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## 2. COHOMOTOPY GROUPS ASSOCIATED WITH ELEMENTS IN $K(B)$

**2.1. Definition of  ${}_{S^1}\alpha_B^*(X, Y)$ .** Let  $B$  be a compact topological space endowed with the trivial  $S^1$ -action. Let  $\mathcal{C}_B$  be the category defined in the following way: the objects of  $\mathcal{C}_B$  are vector bundles over  $B$  of the form

$$\xi = \eta \oplus \xi_0 ,$$

where  $\eta$  is a complex vector bundle endowed with the standard  $S^1$ -action and  $\xi_0$  is a real vector bundle endowed with the trivial  $S^1$ -action; for two objects  $\xi = \eta \oplus \xi_0$ ,  $\xi' = \eta' \oplus \xi'_0$  a morphism  $\nu : \xi \rightarrow \xi'$  is a pair  $(i, \zeta)$  consisting of an  $S^1$ -equivariant bundle embedding  $i = \iota \oplus i_0 : \xi \rightarrow \xi'$  and a complement  $\zeta = \nu \oplus \zeta_0$  of  $i(\xi)$  in  $\xi'$ . Composition of morphisms is defined in a natural way. A morphism  $u = (i, \zeta) : \xi \rightarrow \xi'$  defines a push-forward morphism  $A(u) : A(\xi) \rightarrow A(\xi')$ , where  $A(\xi) := A(\eta) \times A(\xi_0)$  is the automorphism group of  $\xi$ . We obtain in this way a functor  $A : \mathcal{C}_B \rightarrow \mathcal{G}r$ . In the terminology of section 5.1 the pair  $(\mathcal{C}_B, A)$  is a stable category.

Let  $X \rightarrow B, Y \rightarrow B$  be two pointed  $S^1$ -spaces over  $B$ . The assignment

$$\xi \mapsto {}_{S^1}\pi_B^0(X \wedge_B \xi_B^+, Y \wedge_B \xi_B^+)$$

(where  ${}_{S^1}\pi_B^0(X, Y)$  stands for the set of homotopy classes of  $S^1$ -equivariant base point preserving maps over  $B$ ) is functorial with respect to morphisms in  $\mathcal{C}_B$ : for a morphism  $u = (i, \zeta) : \xi \rightarrow \xi'$ , the push-forward class  $u_*([f])$  is defined using  $i \circ f \circ i^{-1}$  on  $i(\xi)$  and  $\text{id}_\zeta$  on its complement  $\zeta$ . Therefore this assignment defines a

functor  ${}_S^1\pi_B^0(X \wedge_B \cdot, Y \wedge_B \cdot) : \mathcal{C}_B \rightarrow \mathcal{S}ets$ . It is not clear at all that an inductive limit of this functor exists, because  $\mathcal{O}b(\mathcal{C}_B)$  is not a small category, so it is not clear that the classical construction of the inductive limit yields a set (see section 5.1).

**Proposition 2.1.** *Let  $\xi = \eta \oplus \xi_0 \in \mathcal{O}b(\mathcal{C}_B)$ ,  $a = (\alpha, a_0) \in A(\xi)$  and  $u = (i, \zeta)$  the standard morphism  $\eta \oplus \xi_0 = \xi \rightarrow \tilde{\xi} := (\eta \oplus \eta) \oplus (\xi_0 \oplus \xi_0)$  defined by  $(y, x) \mapsto ((y, 0), (x, 0))$ . For every  $[f] \in {}_S^1\pi_B^0(X \wedge_B \xi_B^+, Y \wedge_B \xi_B^+)$  one has*

$$u_*(a_*([f])) = u_*([f]) .$$

**Proof:** Identifying  $\tilde{\xi}$  with  $\xi \oplus \xi$  one can write  $u_*(a_*([f])) = [g]$  where  $g$  is the composition

$$(\text{id}_X \wedge_B [a \oplus \text{id}_\xi]_B^+) \circ (f \wedge_B \text{id}_{\xi_B^+}) \circ (\text{id}_X \wedge_B [a^{-1} \oplus \text{id}_\xi]_B^+) : X \wedge_B [\xi \oplus \xi]_B^+ \rightarrow Y \wedge_B [\xi \oplus \xi]_B^+ .$$

Let  $R_t$  be the automorphism of  $\xi \oplus \xi$  defined by the matrix

$$r_t := \begin{pmatrix} \cos(t\frac{\pi}{2}) & -\sin(t\frac{\pi}{2}) \\ \sin(t\frac{\pi}{2}) & \cos(t\frac{\pi}{2}) \end{pmatrix} .$$

For an automorphism  $b$  of  $\xi$  note that  $r_t \circ (b \oplus \text{id}_\xi) \circ r_t^{-1}$  defines a homotopy between  $b \oplus \text{id}_\xi$  and  $\text{id}_\xi \oplus b$ . This shows that  $g$  is homotopic to the map

$$g' := (\text{id}_X \wedge_B [\text{id}_\xi \oplus a]_B^+) \circ (f \wedge_B \text{id}_{\xi_B^+}) \circ (\text{id}_X \wedge_B [\text{id}_\xi \oplus a^{-1}]_B^+) = f \wedge_B \text{id}_{\xi_B^+}$$

which is a representative of the class  $u_*([f])$ .  $\blacksquare$

We define the stable cohomotopy group  ${}_S^1\alpha_B^0(X, Y)$  by

$${}_S^1\alpha_B^0(X, Y) := \varinjlim_{(n, m) \in \mathbb{N}^2} {}_S^1\pi_B^0(X \wedge_B [\underline{\mathbb{C}}^n \oplus \underline{\mathbb{R}}^m]_B^+, Y \wedge_B (\underline{\mathbb{C}}^n \oplus \underline{\mathbb{R}}^m)_B^+) .$$

In this formula and in the rest of the paper we use the notation  $\underline{V}$  for the trivial bundle  $B \times V$  over the base  $B$ . This inductive limit has a natural Abelian group structure (see [CJ] p. 168 for the non-equivariant case).

**Proposition 2.2.** *The functor  ${}_S^1\pi_B^0(X \wedge_B \cdot, Y \wedge_B \cdot) : \mathcal{C}_B \rightarrow \mathcal{S}ets$  admits an inductive limit, which can be identified with  ${}_S^1\alpha_B^0(X, Y)$ .*

**Proof:** Let  $\mathcal{N}^2$  be the small category associated with the ordered set  $(\mathbb{N} \times \mathbb{N}, \leq)$  and consider the functor  $\Theta : \mathcal{N}^2 \rightarrow \mathcal{C}_B$  which assigns to a pair  $(n, m)$  the trivial bundle  $\underline{\mathbb{C}}^n \oplus \underline{\mathbb{R}}^m$  over  $B$ , and to an inequality  $(n, m) \leq (n', m')$  the standard morphism between the corresponding trivial bundles. Using the terminology of section 5.1,  $\mathcal{N}$  is a small filtering category, and  $\Theta$  is a cofinal functor from  $\mathcal{N}$  to the stable category  $(\mathcal{C}_B, A)$ . By definition,  ${}_S^1\alpha_B^0(X, Y)$  is just the limit of the composed functor  ${}_S^1\pi_B^0(X \wedge_B \cdot, Y \wedge_B \cdot) \circ \Theta$ . On the other hand Proposition 2.1 shows that the functor  ${}_S^1\pi_B^0(X \wedge_B \cdot, Y \wedge_B \cdot)$  satisfies the ‘‘trivial stable actions’’ axioms TSA,  $\Theta$ SA. The result follows from Proposition 5.11 in section 5.1.  $\blacksquare$

Note that Proposition 2.2 implicitly yields a canonical map

$$c_\xi : {}_S^1\pi_B^0(X \wedge_B \xi_B^+, Y \wedge_B \xi_B^+) \rightarrow {}_S^1\alpha_B^0(X, Y)$$

for every  $\xi \in \mathcal{O}b(\mathcal{C}_B)$ , such that the system  $(c_\xi)_{\xi \in \mathcal{O}b(\mathcal{C}_B)}$  satisfies the universal property of the inductive limit.

As in the non-equivariant case we put

$${}_S^1\alpha_B^p(X, Y) := {}_S^1\alpha_B^0(X \wedge_B (\mathbb{R}^N)_B^+, Y \wedge_B (\mathbb{R}^{N+p})_B^+) \quad (N, N+p \geq 0).$$

Each  ${}_S^1\alpha_B^p(X, Y)$  is a bimodule over the ring

$${}_S^1\alpha^0(B) := {}_S^1\alpha^0(B_+, S^0) = {}_S^1\alpha_B^0(B_{+B}, B_{+B}),$$

and  ${}_S^1\alpha_B^*(X, Y) := \bigoplus_{p \in \mathbb{Z}} {}_S^1\alpha_B^p(X, Y)$  is a graded bimodule over the graded ring  ${}_S^1\alpha^*(B) = \bigoplus_{S^1}\alpha^p(B)$ , where

$${}_S^1\alpha^p(B) := {}_S^1\alpha^p(B_+, S^0) = {}_S^1\alpha^0(B_+, S^p).$$

Right and left multiplication with elements in  ${}_S^1\alpha^0(B)$  coincide (see [CJ] p. 172).

**Remark 2.3.** *In the special case when  $Y$  is of the form  $Y = \zeta_B^+$  with  $\zeta \in \mathcal{C}_B$ , one has a canonical identification*

$${}_S^1\alpha_B^0(X, \zeta_B^+) = {}_S^1\alpha^0\left(X \wedge_B [\zeta']_B^+ / \infty, V^+\right),$$

where  $\zeta \oplus \zeta' = \underline{V}$ , and  $V$  has the form  $\mathbb{C}^k \oplus \mathbb{R}^l$ . In the terminology of [BF] the latter group is a stable cohomotopy group formed with respect to the universum generated by the  $S^1$ -representations  $\mathbb{C}$  and  $\mathbb{R}$ .

**2.2. The computation of  ${}_S^1\alpha^k(B_+, V^+)$ .** Let  $S^1 \rightarrow O(V)$  be an orthogonal representation of  $S^1$ . Our next goal is the computation of the group  ${}_S^1\alpha^k(B_+, V^+)$  for  $k \geq 0$ . In particular, we obtain explicit descriptions of the positive summands  ${}_S^1\alpha^k(B) = {}_S^1\alpha^k(B_+, [\mathbb{R}^k]^+)$  of the graded ring  ${}_S^1\alpha^*(B)$ .

Replacing  $V$  by  $V \oplus \mathbb{R}^k$ , we can reduce the problem to the case  $k = 0$ . One has

$${}_S^1\alpha^0(B_+, V^+) = \varinjlim_{(n,m) \in \mathbb{N}^2} [B_+ \wedge [\mathbb{C}^n \oplus \mathbb{R}^m]^+, V^+ \wedge [\mathbb{C}^n \oplus \mathbb{R}^m]^+]_0^{S^1},$$

where  $[\cdot, \cdot]_0^{S^1}$  stands for the set of homotopy classes of  $S^1$ -equivariant maps between two pointed  $S^1$ -spaces.

According to Hauschild's splitting theorem (Satz 3.4 in [H]) there is a natural identification

$$[B_+ \wedge [\mathbb{C}^n \oplus \mathbb{R}^m]^+, [V \oplus \mathbb{C}^n \oplus \mathbb{R}^m]^+]_0^{S^1} = \tag{1}$$

$$[B_+ \wedge [\mathbb{R}^m]^+, [V^{S^1}]^+ \wedge [\mathbb{R}^m]^+]_0 \times \left[ B_+ \wedge \left[ \frac{[\mathbb{C}^n \oplus \mathbb{R}^m]^+}{[\mathbb{R}^m]^+} \right], V^+ \wedge [\mathbb{C}^n \oplus \mathbb{R}^m]^+ \right]_0^{S^1}$$

where the projection on the first factor is given by restriction on the fixed point set. There exists a homeomorphism of  $S^1$ -spaces

$$[\mathbb{C}^n \oplus \mathbb{R}^m]^+ / [\mathbb{R}^m]^+ \approx S(\mathbb{C}^n)_+ \wedge S^{m+1}.$$

Indeed, one has

$$\begin{aligned} & [\mathbb{C}^n \oplus \mathbb{R}^m]^+ / [\mathbb{R}^m]^+ \approx S(\mathbb{C}^n \oplus \mathbb{R}^{m+1}) / S(\mathbb{R}^{m+1}) \approx \\ & \approx S(\mathbb{C}^n) \times D(\mathbb{R}^{m+1}) \cup D(\mathbb{C}^n) \times S(\mathbb{R}^{m+1}) / D(\mathbb{C}^n) \times S(\mathbb{R}^{m+1}) \approx S(\mathbb{C}^n)_+ \wedge S^{m+1}. \end{aligned}$$

Using the natural identification

$$B_+ \wedge [S(\mathbb{C}^n)_+ \wedge S^{m+1}] \approx S(\mathbb{C}^n)_+ \wedge [B_+ \wedge S^{m+1}] \approx S(\mathbb{C}^n) \times [B_+ \wedge S^{m+1}] / S(\mathbb{C}^n) \times \{*\}$$

and denoting by  $\tilde{V}_n$  the associated bundle  $S(\mathbb{C}^n) \times_{S^1} V$  over  $\mathbb{P}(\mathbb{C}^n)$  we find

$$\begin{aligned} & \left[ B_+ \wedge \left[ [\mathbb{C}^n \oplus \mathbb{R}^m]^+ / [\mathbb{R}^m]^+ \right], V^+ \wedge [\mathbb{C}^n \oplus \mathbb{R}^m]^+ \right]_0^{S^1} \cong \\ & \cong \left[ S(\mathbb{C}^n) \times [B_+ \wedge S^{m+1}] / S(\mathbb{C}^n) \times \{*\}, V^+ \wedge [\mathbb{C}^n \oplus \mathbb{R}^m]^+ \right]_0^{S^1} \cong \\ & \cong {}_{S^1} \pi_{S(\mathbb{C}^n)}^0 (S(\mathbb{C}^n) \times [B_+ \wedge S^{m+1}], S(\mathbb{C}^n) \times [V \oplus \mathbb{C}^n \oplus \mathbb{R}^m]^+) \cong \\ & \cong \pi_{\mathbb{P}(\mathbb{C}^n)}^0 \left( \mathbb{P}(\mathbb{C}^n) \times [B_+ \wedge S^{m+1}], [\tilde{V}_n \oplus \mathcal{O}_{\mathbb{P}(\mathbb{C}^n)}(1)^{\oplus n} \oplus \mathbb{R}^m]_{\mathbb{P}(\mathbb{C}^n)}^+ \right) \cong \\ & \cong \pi_{\mathbb{P}(\mathbb{C}^n)}^0 \left( [\mathbb{P}(\mathbb{C}^n) \times [B_+ \wedge S^1]] \wedge_{\mathbb{P}(\mathbb{C}^n)} \underline{S}^m, [\tilde{V}_n \oplus \mathcal{O}_{\mathbb{P}(\mathbb{C}^n)}(1)^{\oplus n}]_{\mathbb{P}(\mathbb{C}^n)}^+ \wedge_{\mathbb{P}(\mathbb{C}^n)} \underline{S}^m \right). \end{aligned}$$

The limit over  $m$  of this set is

$$\omega_{\mathbb{P}(\mathbb{C}^n)}^0 \left( \mathbb{P}(\mathbb{C}^n) \times [B_+ \wedge S^1], [\tilde{V}_n \oplus \mathcal{O}_{\mathbb{P}(\mathbb{C}^n)}(1)^{\oplus n}]_{\mathbb{P}(\mathbb{C}^n)}^+ \right).$$

Now note that

$$\tilde{V} \oplus \underline{\mathbb{C}} \oplus T_{\mathbb{P}(\mathbb{C}^n)} \cong \tilde{V}_n \oplus \mathcal{O}_{\mathbb{P}(\mathbb{C}^n)}(1)^{\oplus n}.$$

Therefore, applying the duality isomorphism given in Proposition 12.41 [CJ] to the map  $\pi : \mathbb{P}(\mathbb{C}^n) \rightarrow \{*\}$ , one gets

$$\begin{aligned} \omega_{\mathbb{P}(\mathbb{C}^n)}^0 \left( \mathbb{P}(\mathbb{C}^n) \times [B_+ \wedge S^1], [\tilde{V}_n \oplus \mathcal{O}_{\mathbb{P}(\mathbb{C}^n)}(1)^{\oplus n}]_{\mathbb{P}(\mathbb{C}^n)}^+ \right) & \cong \omega^0(B_+ \wedge S^1, \pi_*([\tilde{V}_n \oplus \underline{\mathbb{C}}]_{\mathbb{P}(\mathbb{C}^n)}^+)) \\ & \simeq \omega^0(B_+ \wedge S^1, T(\tilde{V}_n \oplus \underline{\mathbb{C}})) \simeq \omega^0(B_+ \wedge S^1, T(\tilde{V}_n) \wedge S^2) \cong \omega^0(B_+, T(\tilde{V}_n) \wedge S^1), \end{aligned}$$

where  $T(\cdot)$  stands for the Thom space functor. This shows that

$$\varinjlim_{(n,m) \in \mathbb{N}^2} \left[ B_+ \wedge \left[ [\mathbb{C}^n \oplus \mathbb{R}^m]^+ / [\mathbb{R}^m]^+ \right], V^+ \wedge [\mathbb{C}^n \oplus \mathbb{R}^m]^+ \right]_0^{S^1} \cong \omega^0(B_+, T(ES^1 \times_{S^1} V) \wedge S^1)$$

where  $ES^1 \times_{S^1} V$  is the vector bundle associated with the universal  $S^1$ -bundle  $ES^1 \rightarrow BS^1 = \mathbb{P}^\infty$  and the fiber  $V$ . Using formula (1) we obtain the following

**Proposition 2.4.** *One has a natural group isomorphism*

$${}_{S^1} \alpha^0(B_+, V^+) \cong \omega^0(B_+, [V^{S^1}]^+) \times \omega^0(B_+, T(ES^1 \times_{S^1} V) \wedge S^1). \quad (2)$$

where the projection on the first factor is given by restriction to the fixed point set. In particular

$${}_{S^1} \alpha^k(B) \cong \omega^k(B) \times \omega^k(B_+, \mathbb{P}_+^\infty \wedge S^1).$$

**Remark 2.5.** *The second summand in the decomposition*

$${}_{S^1} \alpha^0(B) \cong \omega^0(B) \times \omega^0(B_+, \mathbb{P}_+^\infty \wedge S^1)$$

is called “the free summand” in [CK]. The projection  ${}_{S^1} \alpha^0(B) \rightarrow \omega^0(B)$  is given by restriction to the fixed point set, hence it is a ring homomorphism. Therefore the free summand  $\omega^0(B_+, \mathbb{P}_+^\infty \wedge S^1)$  is an ideal of  ${}_{S^1} \alpha^0(B)$ , and one has a natural ring isomorphism

$$\omega^0(B) \simeq {}_{S^1} \alpha^0(B) / \omega^0(B_+, \mathbb{P}_+^\infty \wedge S^1).$$

**Corollary 2.6.** *Suppose that  $B$  is a finite CW complex. Restriction to the fixed point set defines an isomorphism*

$$\varinjlim_{N \in \mathbb{N}} {}_{S^1} \alpha^k(B_+, [\mathbb{C}^N]^+) \xrightarrow{\cong} \omega^k(B).$$



**Proof:** Indeed, taking  $V = \mathbb{C}^N \oplus \mathbb{R}^k$ , the second summand in (2) is

$$\omega^0(B_+, T(ES^1 \times_{S^1} [\mathbb{C}^N \oplus \mathbb{R}^{k+1}])) = \varinjlim_{l \in \mathbb{N}} \pi^0(B_+ \wedge [\mathbb{R}^l]^+, T(ES^1 \times_{S^1} [\mathbb{C}^N \oplus \mathbb{R}^{k+1+l}]))$$

We recall that the Thom space of a real vector bundle of rank  $r$  over a CW complex  $X$  admits a CW decomposition consisting of a single 0-dimensional cell and cells of dimension  $\geq r$ . Therefore, for  $N$  sufficiently large any map  $B_+ \wedge [\mathbb{R}^l]^+ \rightarrow T(ES^1 \times_{S^1} [\mathbb{C}^N \oplus \mathbb{R}^{k+1+l}])$  is homotopically trivial. ■

**2.3. The groups  $\alpha^*(x)$  associated with an element  $x \in K(B)$ .** Fix an element  $x \in K(B)$ . We define a category  $\mathcal{T}(x)$  in the following way: the objects of  $\mathcal{T}(x)$  are the presentations of  $x$ . For two such presentations  $(E, F)$ ,  $(E', F')$ , a morphism  $\tau : (E, F) \rightarrow (E', F')$  is a system  $\tau = (i, j, E_1, F_1, k)$  consisting of bundle monomorphisms  $j : E \hookrightarrow E'$ ,  $i : F \hookrightarrow F'$ , complements  $E_1$  and  $F_1$  of  $i(E)$  and  $j(F)$  in  $E'$  and  $F'$  respectively, and an isomorphism  $k : E_1 \rightarrow F_1$ .

To every  $(E, F) \in x$  we associate the graded group  ${}_{S^1} \alpha_B^*(S(E)_{+B}, F_B^+)$ . We claim that a morphism  $\tau : (E, F) \rightarrow (E', F')$  induces a morphism

$$\tau_* : {}_{S^1} \alpha_B^*(S(E)_{+B}, F_B^+) \longrightarrow {}_{S^1} \alpha_B^*(S(E')_{+B}, [F']_B^+).$$

Note first that, for Euclidean or Hermitian vector spaces  $V, W$ , one has a contraction

$$S(V \oplus W) \rightarrow S(V)_+ \wedge W^+$$

induced by the map

$$\begin{aligned} c : S(V \oplus W) &= [S(V) \times D(W)] \cup_{S(V) \times S(W)} [D(V) \times S(W)] \longrightarrow \\ &\longrightarrow S(V) \times D(W) / S(V) \times S(W) \simeq S(V) \times W^+ / S(V) \times \infty_W = S(V)_+ \wedge W^+. \end{aligned}$$

It is useful to have explicit analytic formulae for the contraction map  $c$ . For this it is convenient to parameterize  $S(V \oplus W)$  using polar coordinates on the  $V$ -component. In other words, consider the surjective map

$$\lambda : S(V) \times S^{\geq 0}(\mathbb{R} \oplus W) \rightarrow S(V \oplus W), \quad \lambda(v, r, w) = (rv, w),$$

where  $S^{\geq 0}(\mathbb{R} \oplus W)$  stands for the hemisphere of  $S(\mathbb{R} \oplus W)$  consisting of pairs with positive first component.

$$S^{\geq 0}(\mathbb{R} \oplus W) := \{(r, w) \in \mathbb{R}_{\geq 0} \times W \mid r^2 + \|w\|^2 = 1\}.$$

This map contracts  $S(V) \times \{0\} \times S(W)$  to  $\{0_V\} \times S(W)$  fiberwise over  $S(W)$  and defines an isomorphism between the complements of these subspaces. One can define  $W^+$  in two equivalent ways: as the one-point compactification of  $W$ , and as the quotient  $D(W)/S(W)$ . Accordingly, the contraction maps  $c, c' : S(V \oplus W) \rightarrow S(V)_+ \wedge W^+$  are induced respectively by the maps  $\tilde{c}, \tilde{c}'$  defined on the product  $S(V) \times S^{\geq 0}(\mathbb{R} \oplus W)$  by

$$\tilde{c}(v, r, w) := [v, r^{-1}w], \quad \tilde{c}'(v, r, w) = [v, w].$$

The corresponding formulae for  $c, c'$  are

$$c(v, w) = \begin{cases} \left( \frac{1}{\|v\|} v, \frac{1}{\sqrt{1-\|w\|^2}} w \right) & v \neq 0 \\ * & v = 0 \end{cases}, \quad c'(v, w) = \begin{cases} \left( \frac{1}{\|v\|} v, w \right) & v \neq 0 \\ * & v = 0. \end{cases} \quad (3)$$

To save on notations we will still write  $c, \tilde{c}$  instead of  $c', \tilde{c}'$ .

Therefore, in the presence of a morphism  $\tau = (i, j, E_1, F_1, k) : (E, F) \rightarrow (E', F')$  one gets a map

$$S(E')_{+B} = S(i(E) \oplus E_1)_{+B} \xrightarrow{c} S(i(E))_{+B} \wedge_B (E_1)_B^+,$$

which is well defined up to homotopy (the section  $+_B$  on the left is mapped fiberwise to the distinguished section on the right). We obtain morphisms

$$\begin{aligned} & S^1\alpha_B^*(S(E)_{+B}, F_B^+) \xrightarrow{(i,j)\simeq} S^1\alpha_B^*(S(i(E))_{+B}, j(F)_B^+) = \\ & = S^1\alpha_B^*(S(i(E))_{+B} \wedge_B (F_1)_B^+, j(F)_B^+ \wedge_B (F_1)_B^+) = S^1\alpha_B^*(S(i(E))_{+B} \wedge_B (F_1)_B^+, (F')_B^+) \\ & \xrightarrow{k} S^1\alpha_B^*(S(i(E))_{+B} \wedge_B (E_1)_B^+, (F')_B^+) \xrightarrow{c^*} S^1\alpha_B^*(S(E')_{+B}, (F')_B^+) . \end{aligned}$$

The composition of these maps will be denoted by  $\tau_*$ . One checks that  $\tau_*$  is a morphism of  $S^1\alpha^*(B)$ -modules and, for any two composable morphisms  $\tau, \tau'$ , one has

$$(\tau' \circ \tau)_* = \tau'_* \circ \tau_* .$$

In other words, the assignment  $(E, F) \mapsto S^1\alpha_B^*(S(E)_{+B}, F_B^+)$  is functorial, so it defines a functor  $\alpha_x^* : \mathcal{T}(x) \rightarrow \mathcal{A}b^*$ .

**Example:** Suppose that the stable class  $\varphi \in S^1\alpha_B^0(S(E)_{+B}, F_B^+)$  is represented by an  $S^1$ -equivariant map  $f : S(E) \rightarrow F_B^+$  over  $B$  (or, equivalently, by an  $S^1$ -equivariant map  $S(E)_{+B} \rightarrow F_B^+$  of pointed spaces over  $B$ ). Let  $U$  be a complex vector bundle over  $B$  and let  $\tau$  be the obvious morphism  $(E, F) \rightarrow (E \oplus U, F \oplus U)$ . Then  $f$  defines a map

$$[S(E) \times_B U_B^+] / [S(E) \times_B \infty_U] \longrightarrow F_B^+ \times_B U_B^+ / [F_B^+ \times_B \infty_U]$$

which, composed from the right with the contraction

$$S(E \oplus U) \rightarrow [S(E) \times_B U_B^+] / [S(E) \times_B \infty_U]$$

and from on left with the contraction

$$F_B^+ \times_B U_B^+ / [F_B^+ \times_B \infty_U] \rightarrow F_B^+ \times_B U_B^+ / [F_B^+ \times_B \infty_U \cup \infty_F \times_B U_B^+] = (F \oplus U)_B^+$$

gives an  $S^1$ -equivariant map  $S(E \oplus U) \rightarrow (F \oplus U)_B^+$  over  $B$ . This map represents  $\tau_*(\varphi) \in S^1\alpha_B^0(S(E \oplus U)_{+B}, (F \oplus U)_B^+)$ .

Let  $a \in \text{Aut}(E)$  be a unitary gauge transformation of the bundle  $E$ . Composing with the induced automorphisms  $S(a)$  of the sphere bundles  $S(E)_{+B}$  defines a morphism

$$S^1\alpha_B^*(S(E)_{+B}, F_B^+) \xrightarrow{S(a)^*} S^1\alpha_B^*(S(E)_{+B}, F_B^+) .$$

On the other hand,  $a$  defines an element  $[a_B^+] \in S^1\pi_B^0(E_B^+, E_B^+)$ , whose stable class  $\{a_B^+\}$  is a unit in the ground ring  $S^1\alpha^0(B)$  and defines multiplication automorphisms

$$S^1\alpha_B^*(S(E)_{+B}, F_B^+) \xrightarrow{m(a)} S^1\alpha_B^*(S(E)_{+B}, F_B^+) .$$

Clearly these automorphisms depend only on the homotopy class of  $a$ .

**Proposition 2.7.** *Let  $\varphi \in S^1\alpha^*(S(E)_{+B}, F_B^+)$  and  $a \in \text{Aut}(E)$ . Let  $\tau$  be the obvious morphism  $\tau : (E, F) \rightarrow (E \oplus E, F \oplus E)$ . In  $S^1\alpha^*(S(E \oplus E)_{+B}, [F \oplus E]_B^+)$  is holds*

$$\tau_*(S(a)^*(\varphi)) = \tau_*(m(a)(\varphi)) .$$

**Proof:** For simplicity we prove the statement only in degree 0. We may assume that  $a$  is a unitary automorphism with respect to a Hermitian structure on  $E$ . Suppose that  $\varphi$  is represented by

$$[f] \in {}_{S^1}\pi_B^0(S(E)_{+B} \wedge_B \xi_B^+, F_B^+ \wedge_B \xi_B^+).$$

We will prove that the natural representatives

$$p, q \in {}_{S^1}\text{Map}_B(S(E \oplus E)_{+B} \wedge_B \xi_B^+ \wedge_B E_B^+, (F \oplus E)_B^+ \wedge_B \xi_B^+ \wedge_B E_B^+)$$

of  $\tau_*(S(a)^*([f]))$ ,  $\tau_*(m(a)([f]))$  are homotopic, so they define the same element in

$${}_{S^1}\pi_B^0(S(E \oplus E)_{+B} \wedge_B \xi_B^+ \wedge_B E_B^+, (F \oplus E)_B^+ \wedge_B \xi_B^+ \wedge_B E_B^+).$$

We suppose for simplicity that  $\xi$  is trivial, to save on notations. Consider the contraction map  $c : S(E \oplus E)_{+B} \rightarrow S(E)_{+B} \wedge_B E_B^+$  defined by the first formula in (3), and introduce the maps

$$\Psi, \chi : S(E)_{+B} \wedge_B E_B^+ \wedge_B E_B^+ \longrightarrow F_B^+ \wedge_B E_B^+ \wedge_B E_B^+$$

defined by

$$\Psi := [f \circ S(a)] \wedge_B \text{id}_{E_B^+} \wedge_B \text{id}_{E_B^+}, \quad \chi := f \wedge_B \text{id}_{E_B^+} \wedge_B a_B^+.$$

Using our definitions it is easy to see that  $p = \Psi \circ (c \wedge_B \text{id}_{E_B^+})$ ,  $q = \chi \circ (c \wedge_B \text{id}_{E_B^+})$ . Use the same method as in the proof of Proposition 2.1 (conjugation with the rotations of  $E \oplus E$  defined by the matrices  $r_t$ ) to construct a homotopy

$$\chi = f \wedge_B (\text{id}_E \oplus a)_B^+ \simeq f \wedge_B (a \oplus \text{id}_E)_B^+ = f \wedge_B a_B^+ \wedge_B \text{id}_{E_B^+} := \chi'.$$

It suffices to construct a homotopy between  $\Psi \circ (c \wedge_B \text{id}_{E_B^+})$ , and  $\chi' \circ (c \wedge_B \text{id}_{E_B^+})$ , and for this it suffices to construct a homotopy between the maps  $\Psi_0 \circ c$  and  $\chi'_0 \circ c$ , where

$$\Psi_0 := [f \circ S(a)] \wedge_B \text{id}_{E_B^+} = (f \wedge_B \text{id}_{E_B^+}) \circ (S(a) \wedge_B \text{id}_{E_B^+}),$$

$$\chi'_0 := f \wedge_B a_B^+ = (f \wedge_B \text{id}_{E_B^+}) \circ (\text{id}_{E_B^+} \wedge_B a_B^+).$$

Note that  $(S(a) \wedge_B \text{id}_{E_B^+}) \circ c = c \circ S(a \oplus \text{id}_E)$ , and  $(\text{id}_{S(E)} \wedge_B a_B^+) \circ c = c \circ S(\text{id}_E \oplus a)$ . In these formulae we use the fact that  $a$  is a unitary. On the other hand, using again conjugation with the rotations defined by the matrices  $r_t$ , we see that  $S(a \oplus \text{id}) \simeq S(\text{id}_E \oplus a)$ . Therefore

$$\begin{aligned} \Psi_0 \circ c &= (f \wedge_B \text{id}_{E_B^+}) \circ c \circ S(a \oplus \text{id}) \simeq (f \wedge_B \text{id}_{E_B^+}) \circ c \circ S(\text{id}_E \oplus a) = \\ &= (f \wedge_B \text{id}_{E_B^+}) \circ (\text{id}_{S(E)} \wedge_B a_B^+) \circ c = \chi'_0, \end{aligned}$$

which completes the proof.  $\blacksquare$

A similar statement holds for the action of an automorphism  $b \in \text{Aut}(F)$ . Denote by  $[b_B^+]_*$  the automorphism of  ${}_{S^1}\alpha^*(S(E)_{+B}, F_B^+)$  defined by composition with  $b_B^+$ .

**Proposition 2.8.** *The automorphisms  $[b_B^+]_*$ ,  $m(b)$  coincide on  ${}_{S^1}\alpha^*(S(E)_{+B}, F_B^+)$ .*

The proof uses similar arguments as the proof of Proposition 2.7 but is substantially easier.  $\blacksquare$

An automorphism  $c \in \text{Aut}(U)$  defines a automorphism  $\sigma(c)$  of the graded group  $\alpha^*(S(E \oplus U)_{+B}, [F \oplus U]_B^+)$  defined by  $f \mapsto [\text{id}_F \oplus c]_B^+ \circ f \circ S(\text{id}_E \oplus c)^{-1}$ .

**Corollary 2.9.** *Let  $\tau : (E \oplus U, F \oplus U) \rightarrow (E \oplus U \oplus E \oplus U, F \oplus U \oplus E \oplus U)$  be the natural morphism. Then for any  $\varphi \in \alpha^*(S(E \oplus U)_{+B}, [F \oplus U]_B^+)$  one has*

$$\tau_*(\sigma(c)(\varphi)) = \tau_*(\varphi) .$$

**Proof:** Indeed, one has

$$\tau_* \circ \{[\text{id}_F \oplus c]_B^+\}_* = \tau_* \circ m(c) , \quad \tau_* \circ \{S(\text{id}_E \oplus c)^{-1}\}_* = \tau_* \circ (m(c)^{-1}) .$$

On the other hand the morphism  $\tau_*$  is  ${}_{S^1}\alpha^0(B)$ -linear. ■

Consider now the category  $\mathcal{U}_B$  of all finite rank complex vector bundles over  $B$ . A morphism  $\nu : U \rightarrow U'$  in the category  $\mathcal{U}_B$  is a pair  $(i, U_1)$  consisting of a bundle embedding  $i : U \rightarrow U'$  and a complement  $U_1$  of  $i(U)$  in  $U'$ . The assignment  $U \mapsto {}_{S^1}\alpha_B^*(S(E \oplus U)_{+B}, (F \oplus U)_B^+)$  is functorial with respect to morphisms in  $\mathcal{U}_B$ , so it functor  $\mathfrak{a}_{E,F}^* : \mathcal{U}_B \rightarrow \mathcal{A}b^*$ . Since  $\mathcal{U}_B$  is not a small category, it is not clear whether this functor has an inductive limit (see sections 2.1, 5.1). We put

$$\hat{\alpha}^*(E, F) := \varinjlim_{n \in \mathbb{N}} {}_{S^1}\alpha_B^*(S(E \oplus \mathbb{C}^n)_{+B}, (F \oplus \mathbb{C}^n)_B^+) . \quad (4)$$

**Proposition 2.10.** *The functor  $\mathfrak{a}_{E,F}^*$  admits an inductive limit which can be identified with  $\hat{\alpha}^*(E, F)$ .*

**Proof:** Let  $\mathcal{N}$  be the category associated with the ordered set  $(\mathbb{N}, \leq)$  and let  $\Theta : \mathcal{N} \rightarrow \mathcal{U}_B$  the cofinal functor  $n \mapsto \mathbb{C}^n$  (see section 5.1). By Corollary 2.3, the functor  $\mathfrak{a}_{E,F}^*$  satisfies the trivial stable action axiom  $\Theta$ SA. The result follows now from Proposition 5.11 in section 5.1. ■

In particular one has canonical morphisms  $c_U : {}_{S^1}\alpha_B^*(S(E \oplus U)_{+B}, (F \oplus U)_B^+) \rightarrow \hat{\alpha}^*(E, F)$  for any complex bundle  $U$ , and the system  $(c_U)_U$  is  $\mathfrak{a}_{E,F}^*$ -compatible and satisfies the universal property of the inductive limit. Note that  $\hat{\alpha}^*(E, F)$  has a natural structure of a graded  ${}_{S^1}\alpha^*(B)$  bimodule. By Propositions 2.7 and 2.8 we get:

**Remark 2.11.** *The action of the gauge groups  $\text{Aut}(E \oplus U)$ ,  $\text{Aut}(F \oplus U)$  on  $\hat{\alpha}^*(E, F)$  is induced by the canonical  ${}_{S^1}\alpha^0(B)^\times$ -action defined by its module structure via the morphisms  $\text{Aut}(E \oplus U) \rightarrow {}_{S^1}\alpha^0(B)^\times$ ,  $\text{Aut}(F \oplus U) \rightarrow {}_{S^1}\alpha^0(B)^\times$  defined by  $a \mapsto a_B^+$ .*

A morphism  $\tau = (i, j, E_1, F_1, k) : (E, F) \rightarrow (E', F')$  between two presentations  $(E, F)$ ,  $(E', F')$  of  $x$  induces a sequence a morphisms  $(E \oplus \mathbb{C}^n, F \oplus \mathbb{C}^n) \rightarrow (E' \oplus \mathbb{C}^n, F' \oplus \mathbb{C}^n)$ , so it induces a morphism  $\hat{\tau}_* : \hat{\alpha}^*(E, F) \xrightarrow{\cong} \hat{\alpha}^*(E', F')$ . It is easy to see that  $\hat{\tau}_*$  is an isomorphism: it suffices to note that there exists an isomorphism  $\theta : (E', F') \rightarrow (E \oplus U, F \oplus U)$  (with  $U := E_1$ ) such that  $\theta \circ \tau$  is the standard morphism  $(E, F) \rightarrow (E \oplus U, F \oplus U)$ , and to apply Proposition 2.10. Therefore we obtain a functor  $\hat{\alpha}_x^* : \mathcal{T}(x) \rightarrow \mathcal{A}b^*$  whose associated morphisms  $\hat{\alpha}_x^*(\tau) = \hat{\tau}_*$  are all isomorphisms. According to Proposition 5.8 an inductive limit of this functor exists and can be identified with a quotient of  $\hat{\alpha}^*(E, F)$ , for any fixed presentation  $(E, F)$  of  $x$ . Therefore we can

**Definition 2.12.** *Define*

$$\alpha^*(x) := \varinjlim_{(E,F) \in x} \hat{\alpha}^*(E, F) .$$

**Remark 2.13.** *This inductive limit is also an inductive limit of the functor  $\alpha_x^*$  introduced at the beginning of this section. The existence of the inductive limit of this functor is a non-trivial statement.*

We introduce now the notations

$$\mathbb{A}(E) := \varinjlim_{N \in \mathbb{N}} \text{Aut}(E \oplus \mathbb{C}^N), \quad \mathbb{A}(F) := \varinjlim_{N \in \mathbb{N}} \text{Aut}(F \oplus \mathbb{C}^N), \quad \mathbb{A}(E, F) := \mathbb{A}(E) \times \mathbb{A}(F).$$

Let  $\mathbb{Z}[\mathbb{A}(E, F)]$  be the group ring of this group. By Proposition 5.8 we know that  $\alpha^*(x)$  can be identified with the quotient of  $\hat{\alpha}^*(x)$  by the subgroup  $I[\mathbb{A}(E, F)]\hat{\alpha}^*(x)$ , where  $I[\mathbb{A}(E, F)]$  is the kernel of the canonical morphism  $\mathbb{Z}[\mathbb{A}(E, F)] \rightarrow \mathbb{Z}$  (see section 5.1). But the two groups  $\mathbb{A}(E)$ ,  $\mathbb{A}(F)$  act on  $\hat{\alpha}^*(x)$  via the group morphisms  $l : \mathbb{A}(E) \rightarrow {}_{S^1}\alpha^0(B)^\times$ ,  $r : \mathbb{A}(F) \rightarrow {}_{S^1}\alpha^0(B)^\times$  (see Remark 2.11), so the two actions commute. Let  $\lambda : \mathbb{Z}[\mathbb{A}(E)] \rightarrow {}_{S^1}\alpha^0(B)$ ,  $\rho : \mathbb{Z}[\mathbb{A}(F)] \rightarrow {}_{S^1}\alpha^0(B)$  the ring morphisms associated with the group morphisms  $l$ ,  $r$ . We get

**Remark 2.14.** *For every presentation  $(E, F) \in x$  there is a canonical isomorphism*

$$\alpha(x) \xrightarrow{\cong} \hat{\alpha}^*(E, F) / \lambda(I[\mathbb{A}(E)])\hat{\alpha}^*(E, F) + \rho(I[\mathbb{A}(F)])\hat{\alpha}^*(E, F).$$

In the next section we will see that  $\mathbb{A}(E) \simeq \mathbb{A}(F) \simeq K^{-1}(B)$  and we will identify the images  $\lambda(I[\mathbb{A}(E)])$ ,  $\rho(I[\mathbb{A}(F)])$  of the two ideals in  ${}_{S^1}\alpha^0(B)$  with the image of the ideal  $I[K^{-1}(B)]$  under the ring morphism  $\mathbb{Z}[K^{-1}(B)] \rightarrow {}_{S^1}\alpha^0(B)$  induced by the  $J$ -map  $K^{-1}(B) \rightarrow {}_{S^1}\alpha^0(B)^\times$ .

**2.4. The  $S^1$ -equivariant  $J$ -map and the description of  $\alpha^*(x)$ .** Let  $\pi : E \rightarrow B$  be a Hermitian vector bundle over a compact basis, and  $a, b \in \text{Aut}(E)$  two unitary automorphisms. We define a map

$$\Delta_E(a, b) : S(E)_{+B} \wedge_B \underline{S}^1 \longrightarrow E_B^+$$

in the following way: We use the models

$$S(E)_{+B} \wedge_B \underline{S}^1 \cong S(E) \times [0, 1] / S(E) \times \{0, 1\}, \quad E_B^+ \cong D(E) / {}_B S(E)$$

for the two spaces, and we define

$$\Delta_E(a, b)([e, t]) := \begin{cases} [(1-2t)a(e)] & \text{for } 0 \leq t \leq \frac{1}{2} \\ [(2t-1)b(e)] & \text{for } \frac{1}{2} \leq t \leq 1. \end{cases}$$

Consider the contraction map

$$\mathfrak{c}_E : E_B^+ \longrightarrow S(E)_{+B} \wedge_B \underline{S}^1$$

induced by  $e \mapsto [(\frac{1}{\|e\|}e, \|e\|)]$ . One has

$$\{\Delta_E(a, b)\} = \{b_B^+\} - \{a_B^+\}. \quad (5)$$

**Definition 2.15.** *The  $J$ -homomorphism associated with a Hermitian bundle  $E$  is the morphism  $J_E : \pi_0(\text{Aut}(E)) \rightarrow {}_{S^1}\alpha_B^0(B)^\times$  defined by  $J_E([a]) := \{a_B^+\}$ .*

We introduce the map

$$\Theta_E : \pi_0(\text{Aut}(E)) \longrightarrow {}_{S^1}\alpha_B^{-1}(S(E)_{+B}, E_B^+), \quad \Theta_E([a]) := \{\Delta_E(\text{id}_E, a)\}.$$

Let  $\partial_E : {}_{S^1}\alpha_B^{-1}(S(E)_{+B}, E_B^+) \rightarrow {}_{S^1}\alpha_B^0(E_B^+, E_B^+)$  be the connecting morphism in the long exact cohomotopy sequence:

$$\cdots \rightarrow {}_{S^1}\alpha_B^{-1}(S(E)_{+B}, E_B^+) \xrightarrow{\partial_E} {}_{S^1}\alpha_B^0(E_B^+, E_B^+) \rightarrow {}_{S^1}\alpha_B^0(B_{+B}, E_B^+) \rightarrow \cdots \quad (6)$$

associated with  $E_B^+$  and cofiber sequence

$$S(E)_{+B} \longrightarrow D(E)_{+B} \longrightarrow E_B^+.$$

Since  $\partial_E$  acts by composition with the contraction  $\mathfrak{c}_E$ , we see that the diagram

$$\begin{array}{ccc} \pi_0(\mathrm{Aut}(E)) & \xrightarrow{\Theta_E} & {}_{S^1}\alpha_B^{-1}(S(E)_{+B}, E_B^+) \\ J_E \downarrow & & \downarrow \partial_E \\ {}_{S^1}\alpha^0(B)^\times & \xrightarrow{\cdot - 1} & {}_{S^1}\alpha^0(B) = {}_{S^1}\alpha_B^0(E_B^+, E_B^+) \end{array} \quad (7)$$

is commutative.

**Remark 2.16.** Let  $\omega^0(B_+, \mathbb{P}_+^\infty \wedge S^1) \subset {}_{S^1}\alpha^0(B)$  be the free summand of the ring  ${}_{S^1}\alpha^0(B)$  (see Proposition 2.4). For any  $[a] \in \pi_0(\mathrm{Aut}(E))$  it holds

$$J_E([a]) - 1 \in \omega^0(B_+, \mathbb{P}_+^\infty \wedge S^1) .$$

Indeed,  $\omega^0(B_+, \mathbb{P}_+^\infty \wedge S^1)$  is the kernel of the morphism  $\rho : {}_{S^1}\alpha^0(B) \rightarrow \omega^0(B)$  given by restriction to the fixed point set. Therefore

$$\rho(J_E([a])) = \rho(\{a_B^+\}) = \{(a_B^+)^{S^1}\} = \{\mathrm{id}_{B_+B}\} .$$

■

**Proposition 2.17.** One has

(1)

$$\varinjlim_N \pi_0(\mathrm{Aut}(E \oplus \underline{\mathbb{C}}^N)) = K^{-1}(B)$$

(2) The system of morphisms  $(\partial_{E \oplus \underline{\mathbb{C}}^N})_{N \in \mathbb{N}}$  defines an isomorphism

$$\partial : \varinjlim_N {}_{S^1}\alpha_B^{-1}(S(E \oplus \underline{\mathbb{C}}^N)_{+B}, [E \oplus \underline{\mathbb{C}}^N]_B^+) \longrightarrow \omega^0(B_+, \mathbb{P}_+^\infty \wedge S^1) .$$

**Proof:** For the first statement, note that we have obvious morphisms

$$\varinjlim_N \pi_0(\mathrm{Aut}(\underline{\mathbb{C}}^N)) \rightarrow \varinjlim_N \pi_0(\mathrm{Aut}(E \oplus \underline{\mathbb{C}}^N)) \rightarrow K^{-1}(B) .$$

The right hand morphism is obtained by associating to  $a \in \mathrm{Aut}(E \oplus \underline{\mathbb{C}}^N)$  the bundle over  $B \times S^1$  obtained from  $[E \oplus \underline{\mathbb{C}}^N] \times [0, 1]$  by identifying  $[E \oplus \underline{\mathbb{C}}^N] \times \{0\}$  with  $[E \oplus \underline{\mathbb{C}}^N] \times \{1\}$  via  $a$ . It is well known that the composition of these two morphisms is an isomorphism.

For the second isomorphism, we take the direct limit over  $N$  in the cohomotopy exact sequence (6) associated with  $E \oplus \underline{\mathbb{C}}^N$ . We have

$$\varinjlim_N {}_{S^1}\alpha_B^k([E \oplus \underline{\mathbb{C}}^N]_B^+, [E \oplus \underline{\mathbb{C}}^N]_B^+) = {}_{S^1}\alpha^k(B) .$$

On the other hand, the system of morphisms defined by restricting to the fixed point set (see section 2.2) defines a morphism

$$r^k : \varinjlim_N {}_{S^1}\alpha_B^k(B_{+B}, [E \oplus \underline{\mathbb{C}}^N]_B^+) \rightarrow \omega^k(B_+, S^0) = \omega^k(B) .$$

The composition

$$\begin{aligned} \varinjlim_{N \in \mathbb{N}} {}_{S^1}\alpha^k(B_+, [\mathbb{C}^N]^+) &= \varinjlim_{N \in \mathbb{N}} {}_{S^1}\alpha_B^k(B_{+B}, B \times [\mathbb{C}^N]^+) \rightarrow \\ &\rightarrow \varinjlim_N {}_{S^1}\alpha_B^k(B_{+B}, [E \oplus \underline{\mathbb{C}}^N]_B^+) \rightarrow \omega^k(B) \end{aligned}$$

is an isomorphism by Corollary 2.6, hence  $r^k$  is an isomorphism. The limit of (6) becomes

$${}_{S^1}\alpha^{-1}(B) \xrightarrow{\rho^{-1}} \omega^{-1}(B) \rightarrow \varinjlim_N {}_{S^1}\alpha_B^{-1}(S(E \oplus \underline{\mathbb{C}}^N)_{+B}, [E \oplus \underline{\mathbb{C}}^N]_B^+) \xrightarrow{\partial} {}_{S^1}\alpha^0(B) \xrightarrow{\rho} \omega^0(B)$$

But the map

$$\rho^{-1} : {}_{S^1}\alpha^{-1}(B) = {}_{S^1}\alpha^0(B_+ \wedge S^1) \rightarrow \omega^0(B_+ \wedge S^1) = \omega^{-1}(B)$$

is also induced by restriction to the fixed point set, so it is surjective by Remark 2.5 applied to the basis  $B_+ \wedge S^1$ . Therefore

$$\varinjlim_N {}_{S^1}\alpha_B^{-1}(S(E \oplus \underline{\mathbb{C}}^N)_{+B}, [E \oplus \underline{\mathbb{C}}^N]_B^+) \xrightarrow{\simeq \partial} \ker(\rho) = \omega^0(B_+, \mathbb{P}_+^\infty \wedge S^1) .$$

■

Taking the inductive limit with respect to  $N$  of the diagram (7) written for  $E \oplus \underline{\mathbb{C}}^N$ , we obtain the commutative diagram

$$\begin{array}{ccc} K^{-1}(B) & \xrightarrow{\Theta} & \varinjlim_N {}_{S^1}\alpha_B^{-1}(S(E \oplus \underline{\mathbb{C}}^N)_{+B}, [E \oplus \underline{\mathbb{C}}^N]_B^+) = \hat{\alpha}^{-1}(E, E) \\ J \downarrow & & \simeq \downarrow \partial \\ {}_{S^1}\alpha^0(B)^\times & \xrightarrow{\cdot - 1} & \omega^0(B_+, \mathbb{P}_+^\infty \wedge S^1) \xrightarrow{\iota} {}_{S^1}\alpha^0(B) . \end{array} \quad (8)$$

**Remark 2.18.** *The map  $\iota \circ \partial \circ \Theta : K^{-1}(B) \rightarrow {}_{S^1}\alpha^0(B)$  satisfies the identity*

$$[\iota \circ \partial \circ \Theta](a + b) = [\iota \circ \partial \circ \Theta](a)[\iota \circ \partial \circ \Theta](b) + [\iota \circ \partial \circ \Theta](a) + [\iota \circ \partial \circ \Theta](b) .$$

*This is the “free  $J$ -map” in the terminology of Crabb-Knapp ([CK], p. 88, p.93).*

**Corollary 2.19.** *The map  $J : K^{-1}(B) \rightarrow {}_{S^1}\alpha^0(B)^\times$  is injective.*

**Proof:** It suffices to note that  $\partial \circ \Theta$  is injective by Corollary 2.5 in [CK]. ■

The group morphism  $J$  extends to a ring morphism  $\tilde{J} : \mathbb{Z}[K^{-1}(B)] \rightarrow {}_{S^1}\alpha^0(B)$ .

**Question:** *Does the subgroup*

$$\tilde{J}(I[K^{-1}(B)]) = \langle \{J(u) - 1 \mid u \in K^{-1}(B)\} \rangle = \langle \text{im}(\partial \circ \Theta) \rangle \subset \omega^0(B_+, \mathbb{P}_+^\infty \wedge S^1)$$

*coincide with the free summand  $\omega^0(B_+, \mathbb{P}_+^\infty \wedge S^1)$ ?*

We come back to our problem: the description of  $\alpha^*(x)$ : Using Remarks 2.11 and 2.14 one gets the following descriptions of  $\alpha^*(x)$ .

**Proposition 2.20.** *For every presentation  $(E, F) \in x$  there exist canonical isomorphisms*

$$\alpha^*(x) \simeq \bar{\alpha}^*(E, F) \simeq \hat{\alpha}^*(E, F) / \tilde{J}(I[K^{-1}(B)]) \hat{\alpha}^*(E, F) .$$

*Since  $\tilde{J}(I[K^{-1}(B)]) \subset \omega^0(B_+, \mathbb{P}_+^\infty \wedge S^1)$ , which is an ideal of  ${}_{S^1}\alpha^0(B)$ , we get epimorphisms*

$$\alpha^*(x) \longrightarrow \hat{\alpha}^*(E, F) / \omega^0(B_+, \mathbb{P}_+^\infty \wedge S^1) \cdot \hat{\alpha}^*(E, F) .$$

**2.5. Stabilization.** In this section we will show that the morphism

$$\tau_* : {}_{S^1}\alpha^k(S(E)_{+B}, F_B^+) \rightarrow {}_{S^1}\alpha^k(S(E')_{+B}, [F']_B^+) \quad (9)$$

associated with a morphism  $\tau : (E, F) \rightarrow (E', F')$  in the category  $\mathcal{T}(x)$  is an isomorphism as soon as the rank  $f$  of  $F$  is sufficiently large. In other words, for fixed  $k$ , the groups  $\alpha^k(x)$  can be computed using only presentations  $(E, F)$  with a priori bounded ranks.

**Proposition 2.21.** *Suppose that  $B$  is a finite CW complex. The stabilization morphism (9) is an isomorphism for  $2f \geq \dim(B) - k$ .*

**Proof:** A morphism  $\tau$  defines a bundle  $U$  and isomorphisms  $E' = E \oplus U$ ,  $F' = F \oplus U$ . The long exact sequence associated with this cofiber sequence over  $B$

$$S(U)_{+B} \longrightarrow SE'_{+B} \xrightarrow{c} S(E)_{+B} \wedge_B U_B^+$$

and the target space  $[F']_B^+$  contains the segment

$$\rightarrow {}_{S^1}\alpha_B^{k-1}(S(U)_{+B}, [F']_B^+) \xrightarrow{\partial} {}_{S^1}\alpha_B^k(S(E)_{+B} \wedge_B U_B^+, [F']_B^+) \xrightarrow{c^*} {}_{S^1}\alpha_B^k(SE'_{+B}, [F']_B^+).$$

The morphism  $\tau_*$  is defined by  $c^*$  via the identification  ${}_{S^1}\alpha_B^k(S(E)_{+B}, F_B^+) = {}_{S^1}\alpha_B^k(S(E)_{+B} \wedge_B U_B^+, [F']_B^+)$ , so it is an isomorphism as soon as

$${}_{S^1}\alpha_B^{k-1}(S(U)_{+B}, [F']_B^+) = {}_{S^1}\alpha_B^k(S(U)_{+B}, [F']_B^+) = 0.$$

Suppose for simplicity  $k \geq 0$ . A class  $u \in {}_{S^1}\alpha_B^k(S(U)_{+B}, [F']_B^+)$  is represented by an  $S^1$ -equivariant pointed map over  $B$

$$\varphi : S(U)_{+B} \wedge_B \xi_B^+ = S(U) \times_B \xi^+ /_B S(U) \times_B \infty_\xi \longrightarrow [F' \oplus \mathbb{R}^k \oplus \xi]_B^+$$

where  $\xi = \eta \oplus \xi_0$  is the sum of a complex and a real vector bundle. We may suppose that  $\xi_0$  is an oriented bundle, so that all our bundles become oriented bundles. We will prove that any such map is homotopic to the map  $\varphi_\infty$  which maps the left hand space fiberwise onto the section  $\infty_{F' \oplus \mathbb{R}^k \oplus \xi}$ . Denote by  $q : \mathbb{P}(U) \rightarrow B$  the bundle projection and put

$$\tilde{F}' := q^*(F')(1), \quad \tilde{\xi} := q^*(\eta)(1) \oplus q^*(\xi_0).$$

A map  $\varphi$  as above induces a pointed bundle map  $\tilde{\varphi} : \tilde{\xi}_{\mathbb{P}(U)}^+ \longrightarrow [\tilde{F}' \oplus \mathbb{R}^k \oplus \tilde{\xi}]_{\mathbb{P}(U)}^+$  over  $\mathbb{P}(U)$ , and the assignment  $\varphi \mapsto \tilde{\varphi}$  is a bijection. But by Corollary 5.15 in section 5.2, any such pointed bundle map is homotopic to the fiberwise constant bundle map as soon as  $\dim_{\mathbb{R}}(\mathbb{P}(U)) + \text{rk}(\tilde{\xi}) < \text{rk}_{\mathbb{R}}(\tilde{F}') + k + \text{rk}(\tilde{\xi})$ . This condition is equivalent to  $2f > \dim(B) - k - 2$ . Similarly, we will have  ${}_{S^1}\alpha_B^{k-1}(S(U)_{+B}, [F']_B^+) = 0$  as soon as  $2f > \dim(B) - k - 1$ . ■

**2.6. The cohomotopy Euler class of an element in  $K(B)$ .** Let  $x \in K(B)$  and consider a presentation  $(E, F) \in x$ . The map  $o_{(E,F)} : S(E)_{+B} \rightarrow F_B^+$  which sends the section  $+_B$  of  $S(E)_{+B}$  to the infinity section of  $F_B^+$  and maps any point  $e_b \in S(E_b)$  to  $0_b$  is an  $S^1$ -equivariant map of pointed spaces over  $B$ , hence it defines an element  $\{o_{(E,F)}\} \in {}_{S^1}\alpha_B^0(S(E)_{+B}, F_B^+)$ .

One has a canonical isomorphism (see [CJ] Proposition 12.40)

$${}_{S^1}\alpha_B^0(S(E)_{+B}, F_B^+) \simeq {}_{S^1}\alpha_{S(E)}^0(S(E)_{+S(E)}, \pi^*(F)_{S(E)}^+),$$



where  $\pi : S(E) \rightarrow B$  is the obvious projection. Under this isomorphism the class  $[o_{(E,F)}]$  maps to the equivariant Euler class of the bundle  $\pi^*(F)$  over  $S(E)$ . This class is the pull-back of the equivariant Euler class  $\gamma(F) \in {}_{S^1}\alpha^0(B_{+B}, F_B^+)$  of the bundle  $F$  under the projection  $S(E)_{+S(E)} \rightarrow B_{+B}$ .

For any morphism  $\tau = (i, j, E_1, F_1, k) : (E, F) \rightarrow (E', F')$  in the category  $\mathcal{T}(x)$  one has  $\tau_*(\{o_{(E,F)}\}) = \{o_{(E',F')}\}$ . Therefore the assignment  $(E, F) \mapsto -\{o_{(E,F)}\}$  defines a *tautological element*  $\gamma(x) \in \alpha^*(x)$ . This element will be called the equivariant cohomotopy Euler class of  $x$ .

### 3. COHOMOTOPY INVARIANTS ASSOCIATED WITH CERTAIN NON-LINEAR MAPS BETWEEN HILBERT BUNDLES

**3.1. The cylinder construction.** Let  $(E, F)$  be a pair of Hermitian vector bundles over a compact basis  $B$ . Let  $V, W$  be Euclidean vector spaces, and let  $\mu : E \times V \rightarrow [F \times W]_B^+$  be an  $S^1$ -equivariant map over  $B$ . We suppose that  $\mu$  is fiberwise differentiable and its fiberwise differential is continuous on  $E \times V$ . The equivariance property implies that

$$\mu(0^E \times V) \subset [0^F \times W]_B^+. \quad (10)$$

We assume that  $\mu$  has the following properties:

**P1:** (properness) There exist positive constants  $c, C$  such that  $\|\mu(e, v)\| > c$  for all pairs  $(e, v) \in E \times V$  with  $\|(e, v)\| \geq C$ .

**P2:** (restriction to the  $S^1$ -fixed point set)

(1) There exists a direct sum decomposition  $W = H \oplus W_0$  such that

$$\mu(0_y^E, v) = h(y) + l(v), \quad \forall y \in B, \quad \forall v \in V,$$

where  $l : V \xrightarrow{\cong} W_0 \subset W$  is a linear isomorphism, which does not depend on  $y$ , and  $h : B \rightarrow H$  is a continuous map.

(2) The map  $h$  does not vanish on  $B$ . Therefore there exists  $\varepsilon_0 > 0$  such that

$$\|h(y)\| = \|\mathbb{P}_H(\mu(0_y^E, v))\| \geq \varepsilon_0 \quad \forall (y, v) \in B \times V. \quad (11)$$

We fix an orientation  $\mathcal{o}$  of  $H$ , and set  $b := \dim(H)$ . Choose numbers  $R \geq C$  and  $\varepsilon \leq \min(c, \varepsilon_0)$ . The restriction  $\mu_R$  of  $\mu$  to  $D_R(E) \times D_R(V)$  satisfies

$$\|\mu(e, v)\| \geq \varepsilon \quad \forall (e, v) \in \partial[D_R(E) \times D_R(V)] \cup [0^E \times D_R(V)].$$

Therefore,  $\mu_R$  defines an  $S^1$ -equivariant morphism of pairs over  $B$

$$\begin{aligned} \mu_{R,\varepsilon} : (D_R(E) \times D_R(V), \partial[D_R(E) \times D_R(V)] \cup [0^E \times D_R(V)]) &\longrightarrow \\ &\longrightarrow \left( [F \times W]_B^+, [F \times W]_B^+ \setminus \mathring{D}_\varepsilon(F \times W) \right). \end{aligned}$$

The first space  $D_R(E) \times D_R(V)$  of the pair on which  $\mu_{R,\varepsilon}$  is defined can be geometrically regarded as a ‘‘cylinder bundle’’ over  $B$ , whose base is the complex disk bundle  $D(E)$ ; the second space of this pair is the union of the boundary of this cylinder bundle with the central kernel  $0^E \times D_R(V)$ . Using polar coordinates in  $D_R(E)$  we obtain a map  $S(E) \times [0, R] \rightarrow D_R(E)$ , hence a map

$$\rho : S(E) \times [0, R] \times D_R(V) = S(E) \times D_R(\mathbb{R} \oplus V) \rightarrow D_R(E) \times D_R(V)$$

which maps

$$[S(E) \times \{0, R\} \times D_R(V)] \cup [S(E) \times [0, R] \times S_R(V)] = S(E) \times S_R(\mathbb{R} \oplus V)$$

onto the the second component of the pair on which  $\mu_{R,\varepsilon}$  is defined. Here we used suitable models  $D(\mathbb{R} \oplus V)$ ,  $S(\mathbb{R} \oplus V)$  for the disc and the sphere in  $\mathbb{R} \oplus V$ . Therefore, composing  $\mu_{R,\varepsilon}$  with  $\rho$  we get a an  $S^1$ -equivariant map of pairs over  $B$

$$(S(E) \times [0, R] \times D_R(V), S(E) \times (\{0, R\} \times D_R(V) \cup [0, R] \times S_R(V))) = \\ (S(E) \times D_R(\mathbb{R} \oplus V), S(E) \times S_R(\mathbb{R} \oplus V)) \rightarrow ([F \times W]_B^+, [F \times W]_B^+ \setminus \mathring{D}_\varepsilon(F \times W))$$

which we denote by the same symbol  $\mu_{R,\varepsilon}$ . Collapsing fiberwise over  $B$  the second terms of the two pairs, and composing with the natural isomorphism

$$[F \times W]_B^+ /_B [F \times W]_B^+ \setminus \mathring{D}_\varepsilon(F \times W) \simeq [F \times W]_B^+,$$

one gets an  $S^1$ -equivariant map of pointed spaces over  $B$

$$\mu_{R,\varepsilon} : S(E) \times [\mathbb{R} \oplus V]^+ /_B S(E) \times \{\infty\} = S(E)_{+B} \wedge_B [B \times (\mathbb{R} \oplus V)]_B^+ \longrightarrow [F \times W]_B^+.$$

Using the isomorphism  $l : V \xrightarrow{\cong} W_0$  and an orientation preserving isomorphism  $\mathbb{R}^b \simeq H$ , we obtain an element

$$\{\mu\} \in S^1 \alpha_B^{b-1}(S(E)_{+B}, F_B^+),$$

which is obviously independent of the choice of the pair  $(R, \varepsilon)$ . This element will be called the *cohomotopy invariant* of  $\mu$ .

### 3.2. General properties of the invariant $\{\mu\}$ .

#### 1. A vanishing property.

Let  $\mu : E \times V \rightarrow F \times W$  a map satisfying **P1**, **P2**.

**Proposition 3.1.** *If  $\mu|_{D_C(E) \times D_C(V)}$  is nowhere vanishing, then  $\{\mu\} = 0$ .*

**Proof:** We take  $\varepsilon \leq \inf\{\|\mu(e, v)\| \mid \|e\| \leq C, \|v\| \leq C\}$ , and we note that the  $\{[F \times W]_B^+ /_B \{[F \times W]_B^+ \setminus \mathring{D}_\varepsilon(F \times W)\}\}$ -valued pointed map induced by  $\mu_{R,\varepsilon}$  will be fiberwise constant.  $\blacksquare$

#### 2. Homotopy invariance.

Let  $\mu', \mu'' : E \times V \rightarrow [F \times W]_B^+$  two maps satisfying properties **P1**, **P2** with constants  $C', c', \varepsilon'_0$ , and  $C'', c'', \varepsilon''_0$ . We suppose that the property **P2** of the two maps holds for the same decomposition  $W = H \oplus W_0$  of  $W$  and for the same isomorphism  $l : V \rightarrow W_0$ . We introduce the notations

$$\tilde{B} := B \times [0, 1], \quad \tilde{E} := E \times [0, 1] = p_B^*(E), \quad \tilde{F} := F \times [0, 1] = p_B^*(F).$$

**Proposition 3.2.** *Suppose there exists  $C \geq \max(C', C'')$  and a continuous  $S^1$ -equivariant map  $\tilde{\mu} : D_C(\tilde{E}) \times D_C(V) \rightarrow [\tilde{F} \times W]_B^+$  over  $\tilde{B}$  whose restriction to*

$$\partial [D_C(\tilde{E}) \times D_C(V)] \cup [0^{\tilde{E}} \times D_R(V)]$$

*is nowhere vanishing. Then  $\{\mu'\} = \{\mu''\}$  in  $S^1 \alpha_B^{b-1}(S(E)_{+B}, F_B^+)$ .*

**Proof:** The stable classes  $\{\mu'\}$ ,  $\{\mu''\}$  can be computed using the the cylinder  $D_C(\tilde{E}) \times D_C(V)$  and taking

$$\varepsilon \leq \min \left( \varepsilon'_0, \varepsilon''_0, c', c'', \inf \left\{ \|\tilde{\mu}(y)\| \mid y \in \partial[D_C(\tilde{E}) \times D_C(V)] \cup [0^{\tilde{E}} \times D_R(V)] \right\} \right)$$

Applying the cylinder construction with parameters  $C, \varepsilon$  to the map  $\tilde{\mu}$  we obtain a homotopy between the corresponding representatives of the classes  $\{\mu'\}$ ,  $\{\mu''\}$ . ■

### 3. Invariance under $\wedge \text{id}$ .

Let  $\mu : E \times V \rightarrow F \times W$  be a map satisfying the properties **P1**, **P2** with constants  $C, c, \varepsilon_0$ . Let  $U$  be a complex bundle on  $B$ . Replace  $E, F$  by  $E \oplus U, F \oplus U$ , and let  $\tilde{\mu}$  be the map defined by  $\tilde{\mu}(e, u, v) := \iota \circ [\mu(e, v) \wedge u]$ , where  $\iota$  is the obvious identification  $\iota : [F \times W]_B^+ \wedge U_B^+ \rightarrow [(F \oplus U) \times W]_B^+$ . The map  $\tilde{\mu}$  also satisfies properties **P1**, **P2**. The second property is obvious. The first is satisfied with constants  $C, \gamma$ , where  $0 < \gamma \leq c$  is sufficiently small. Indeed, the property **P1** for  $\mu$  implies that there exists  $\alpha > 0$  such that  $\|\mu(e, v)\| > c$  for every  $(e, v) \in E \times V$  with  $\|(e, v)\| \geq C - \alpha$ . For a point  $(e, u, v) \in (E \oplus U) \times V$  with  $\|(e, u, v)\| \geq C$  one has either  $\|(e, v)\| \geq C - \alpha$  or  $\|u\| \geq \alpha$ . In the first case one gets  $\|\tilde{\mu}(e, u, v)\| \geq c$ , and in the second it follows  $\|\tilde{\mu}(e, u, v)\| \geq \alpha$ .

**Proposition 3.3.** *Let  $\tau$  be the obvious morphism  $(E, F) \rightarrow (E \oplus U, F \oplus U)$ . Then*

$$\{\tilde{\mu}\} = \tau_*(\{\mu\}) .$$

**Proof:** By definition,  $\{\tilde{\mu}\}$  is defined by the map of pairs

$$(S(E \oplus U) \times [0, R] \times D_R(V) , S(E \oplus U) \times (\{0, R\} \times D_R(V) \cup [0, R] \times S_R(V))) \rightarrow \\ \xrightarrow{\tilde{\mu}_{R, \varepsilon}} \left( [(F \oplus U) \times W]_B^+ , [(F \oplus U) \times W]_B^+ \setminus \mathring{D}_\varepsilon[(F \oplus U) \times W] \right) ,$$

where  $R \geq C$ . According to the computation in the Example (section 2.1), a representative of  $\tau_*(\mu)$  can be obtained from

$$(S(E) \times [0, R] \times D_R(V) , S(E) \times (\{0, R\} \times D_R(V) \cup [0, R] \times S_R(V))) \longrightarrow \\ \xrightarrow{\mu_{R, \varepsilon}} \left( [F \times W]_B^+ , [F \times W]_B^+ \setminus \mathring{D}_\varepsilon(F \times W) \right)$$

in the following way: Taking fiber product over  $B$  with  $\text{id}_{U_B^+} : U_B^+ \rightarrow U_B^+$ , we get a map

$$S(E) \times_B U_B^+ \times [0, R] \times D_R(V) \longrightarrow [(F \oplus U) \times W]_B^+$$

which sends  $S(E) \times_B (U_B^+ \setminus \mathring{D}(U)) \times [0, R] \times D_R(V)$  to the compact subspace  $\{[(F \oplus U) \times W]_B^+\}_{\|u\| \geq 1}$  of  $\{[(F \oplus U) \times W]_B^+\}$  defined by the inequality  $\|u\| \geq 1$ . Therefore we obtain a map

$$\left[ S(E) \times_B U_B^+ / S(E) \times_B (U_B^+ \setminus \mathring{D}(U)) \right] \times [0, R] \times D_R(V) \longrightarrow \\ \longrightarrow [(F \oplus U) \times W]_B^+ / \{[(F \oplus U) \times W]_B^+\}_{\|u\| \geq 1}$$

which maps

$$\left[ S(E) \times_B U_B^+ / S(E) \times_B (U_B^+ \setminus \mathring{D}(U)) \right] \times (\{0, R\} \times D_R(V) \cup [0, R] \times S_R(V))$$

to the image of the subspace  $\{[(F \oplus U) \times W]_B^+ \}_{\|(f,w)\| \geq \varepsilon} \subset [(F \oplus U) \times W]_B^+$  defined by the inequality  $\|(f,w)\| \geq \varepsilon$ . We compose this map with  $c \times \text{id}_{[0,R]} \times \text{id}_{D_R(V)}$ , where  $c$  is the contraction map

$$S(E \oplus U) \rightarrow S(E) \times_B U_B^+ / S(E) \times_B (U_B^+ \setminus \mathring{D}(U)), \quad (e, u) \mapsto \left( \frac{1}{\|e\|} e, u \right),$$

and get a map

$$\begin{aligned} & S(E \oplus U) \times [0, R] \times D_R(V) / S(E \oplus U) \times (\{0, R\} \times D_R(V) \cup [0, R] \times S_R(V)) \longrightarrow \\ & \longrightarrow [(F \oplus U) \times W]_B^+ / \{[(F \oplus U) \times W]_B^+ \}_{\|u\| \geq 1} \cup \{[(F \oplus U) \times W]_B^+ \}_{\|(f,w)\| \geq \varepsilon} \longrightarrow \\ & \longrightarrow [(F \oplus U) \times W]_B^+ / [(F \oplus U) \times W]_B^+ \setminus \mathring{D}_\varepsilon((F \oplus U) \times W) \simeq [(F \oplus U) \times W]_B^+. \end{aligned}$$

which represents  $\tau_*(\mu_{R,\varepsilon})$ .

As in section 2.1 we parameterize  $S(E \oplus U)$  using polar coordinates on the  $E$ -component; this yields a map

$$\lambda : S(E) \times S^{\geq 0}(\mathbb{R} \oplus U) \rightarrow S(E \oplus U), \quad \lambda(e, r, u) = (re, u).$$

The two representatives of  $\tilde{\mu}_{R,\varepsilon}$  and  $\tau_*(\mu_{R,\varepsilon})$  are induced by the maps

$$\begin{aligned} \alpha : S(E) \times_B S^{\geq 0}(\mathbb{R} \oplus U) \times [0, R] \times D_R(V) &\longrightarrow (F \times W)_B^+ \wedge_B U_B^+, \\ \beta : S(E) \times_B S^{\geq 0}(\mathbb{R} \oplus U) \times [0, R] \times D_R(V) &\longrightarrow (F \times W)_B^+ \wedge_B U_B^+ \end{aligned}$$

defined by

$$\alpha(e, r, u, \rho, v) = [\mu(\rho re, v), \rho u], \quad \beta(e, r, u, \rho, v) = [\mu(\rho e, v), u].$$

More precisely composing these maps with  $\iota \circ p$ , where  $p$  is the projection

$$p : [(F \oplus U) \times W]_B^+ \rightarrow [(F \oplus U) \times W]_B^+ / [(F \oplus U) \times W]_B^+ \setminus \mathring{D}_\varepsilon((F \oplus U) \times W),$$

we get maps which descend to representatives of  $\tilde{\mu}_{R,\varepsilon}$  and  $\tau_*(\mu_{R,\varepsilon})$  respectively. Consider the following natural homotopy  $\gamma$  between  $\alpha$  and  $\beta$ , defined by:

$$\gamma_t(e, r, u, \rho, v) = [\mu(\rho((1-t)r + t)e, v), ((1-t)\rho + t)u]$$

We will see that (taking a smaller  $\varepsilon$  if necessary) the homotopy  $\iota \circ p \circ \gamma$  descends from  $[0, 1] \times S(E) \times_B S^{\geq 0}(\mathbb{R} \oplus U) \times [0, R] \times D_R(V)$  to

$$[0, 1] \times \left[ S(E \oplus U) \times [0, R] \times D_R(V) / S(E \oplus U) \times (\{0, R\} \times D_R(V) \cup [0, R] \times S_R(V)) \right]$$

and defines a homotopy between the two representatives of  $\tilde{\mu}_{R,\varepsilon}$  and  $\tau_*(\mu_{R,\varepsilon})$ . In order to prove this, we have to control the restrictions of  $\gamma$  to the loci defined by  $\|v\| = R$ ,  $\rho = 0$ ,  $\rho = R$  and  $r = 0$ .

**Claim:** For any sufficiently large  $R \geq C$  there exists  $\varepsilon_R > 0$  such that for any positive  $\varepsilon < \varepsilon_R$  it holds

- (i) The maps  $\gamma_t(\cdot, \cdot, \cdot, \cdot, v)$  (for  $\|v\| = R$ ),  $\gamma_t(\cdot, \cdot, \cdot, 0, \cdot)$ , and  $\gamma_t(\cdot, \cdot, \cdot, R, \cdot)$  take all values in the subspace  $[(F \times W) \oplus U]_B^+ \setminus \mathring{D}_\varepsilon((F \times W) \oplus U)$  of  $[(F \times W) \oplus U]_B^+$ .

- (ii)  $\gamma_t(e, 0, u, \rho, v)$  is constant with respect to  $u \in S(U)$  modulo the subspace  $[(F \times W) \oplus U]_B^+ \setminus \mathring{D}_\varepsilon((F \times W) \oplus U)$  of  $[(F \times W) \oplus U]_B^+$  (recall that the map  $S(E) \times S^{\geq 0}(\mathbb{R} \oplus U) \rightarrow S(E \oplus U)$  contracts  $S(E) \times \{0\} \times_B S(U)$  to  $S(E) \times_B 0_U$  fiberwise over  $S(E)$ ).

**Proof of the Claim:** First of all note that  $\|\gamma_t(e, r, t, u, v)\| \geq \varepsilon_0$  for  $\|v\| = R$  by **P1**. Second, the restrictions of  $\gamma_t$  to the loci defined by  $\rho = 0$ ,  $\rho = R$  and  $r = 0$  are  $\gamma_t(e, r, u, 0, v) = (\mu(0, v), tu)$ ,  $\gamma_t(e, r, u, R, v) = (\mu(R((1-t)r+t)e, v), ((1-t)R+t)u)$ ,  $\gamma_t(e, 0, u, \rho, v) = (\mu(\rho te, v), ((1-t)\rho + t)u)$  for  $\|u\| = 1$ .

Therefore, using the second property **P2** of the map  $\mu$ , we get

$$\|\gamma_t(e, r, u, 0, v)\| \geq \varepsilon_0 \quad \forall (t, e, r, u, v) \in [0, 1] \times S(E) \times S^{\geq 0}(\mathbb{R} \oplus U) \times D_R(V). \quad (12)$$

In order to bound  $\|\gamma_t(e, r, u, R, v)\|$  from below, suppose first that  $\|u\| \geq \frac{\sqrt{2}}{2}$ . In this case

$$\|(1-t)R + t)u\| \geq \frac{\sqrt{2}}{2} \min(1, R) \geq \frac{\sqrt{2}}{2} \min(1, C).$$

When  $\|u\| \leq \frac{\sqrt{2}}{2}$ , we must have  $r \geq \frac{\sqrt{2}}{2}$ , hence

$$\|R((1-t)r + t)e\| = \|R((1-t)r + t)\| \geq R \min\left(\frac{\sqrt{2}}{2}, 1\right),$$

so, by the first property of the map  $\mu$ , one has  $\|\mu(R((1-t)r + t)e, v)\| \geq c$  if we choose  $R \geq \sqrt{2}C$ . Therefore, for any  $R \geq \sqrt{2}C$  one obtains

$$\|\gamma_t(e, r, u, R, v)\| \geq \min\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}C, c\right), \quad (13)$$

for all  $(t, e, r, u, v) \in [0, 1] \times S(E) \times S^{\geq 0}(\mathbb{R} \oplus U) \times D_R(V)$ .

In order to complete the proof of Proposition 3.3 it suffices to prove claim (ii). This follows from Lemma 3.4 below.  $\blacksquare$

**Lemma 3.4.** *For ever  $R > 0$  there exists  $\eta_R > 0$  such that*

$$\|\gamma_t(e, 0, u, \rho, v)\| \geq \eta_R \quad \forall (t, e, u, \rho, v) \in [0, 1] \times S(E) \times S(U) \times [0, R] \times D_R(V).$$

**Proof:** Since  $\|\mu(0, v)\| \geq \varepsilon_0$  for any  $v \in V$ , there exists  $\sigma_R > 0$  such that  $\|\mu(e', v)\| \geq \frac{\varepsilon_0}{2}$  for all  $(e', v) \in D_{\sigma_R}(E) \times D_R(V)$ . We may suppose  $\sigma_R \leq 1$ . Fix  $(e, v) \in S(E) \times D_R(V)$  and  $(t, \rho) \in [0, 1] \times [0, R]$ .

Case 1.  $t\rho \leq \sigma_R$ . In this case one has  $\|\rho te\| \leq \sigma_R$ , hence  $\|\mu(\rho te, v)\| \geq \frac{\varepsilon_0}{2}$ .

Case 2.  $t\rho > \sigma_R$ .

a) When  $\rho \leq 1$  we have

$$(1-t)\rho + t = \rho - t\rho + t = \rho + t(1-\rho) \geq \rho + \frac{\sigma_R}{\rho}(1-\rho) = \rho + \frac{\sigma_R}{\rho} - \sigma_R \geq 2\sqrt{\sigma_R} - \sigma_R = \sqrt{\sigma_R}(2 - \sqrt{\sigma_R}) \geq \sqrt{\sigma_R}.$$

b) When  $\rho \geq 1$  we have  $(1-t)\rho + t \geq 1$ .

Therefore, in all case we have a bound from below for one of the components of  $\gamma_t(e, 0, u, \rho, v)$ :

$$\|\mu(\rho te, v)\| \geq \frac{\varepsilon_0}{2} \quad \text{or} \quad \|((1-t)\rho + t)u\| \geq \min(1, \sqrt{\sigma_R}). \quad (14)$$

■

A similar invariance property also holds if one enlarges the real summands  $V$ ,  $W$ , but in this case the proof is obvious. Putting together the two properties we get the following invariance property:

**Proposition 3.5.** *Let  $\mu : E \times V \rightarrow F \times W$  be a map satisfying the properties **P1**, **P2** with constants  $C$ ,  $c$ ,  $\varepsilon_0$  and maps  $l : V \rightarrow W_0$ ,  $h : B \rightarrow H$ . Let  $a : E' \rightarrow F'$  be an isomorphism of complex vector bundles over  $B$  and  $b : V' \rightarrow W'$  an isomorphism of real vector spaces. Let  $\tilde{E} := E \oplus E'$ ,  $\tilde{F} := F \oplus F'$ ,  $\tilde{V} := V \oplus V'$ ,  $\tilde{W} := W \oplus W'$ , and put*

$$\tilde{\mu}(e, e', v, v') = \iota[\mu(e, v) \wedge_B (a(e'), b(v'))] ,$$

where  $\iota$  is the obvious identification

$$\iota : [F \times W]_B^+ \wedge_B (F' \times W')_B^+ \rightarrow [(F \oplus F') \times (W \oplus W')]_B^+ .$$

Then

- (1)  $\tilde{\mu}$  satisfies again the properties **P1**, **P2** with constants  $C$ ,  $\gamma$  (for sufficiently small  $0 < \gamma < c$ ),  $\varepsilon_0$  and maps  $l \oplus b$ ,  $h$ .
- (2)  $\{\tilde{\mu}\} = \tau_*(\{\mu\})$ , where  $\tau$  denotes the obvious morphism  $(E, F) \rightarrow (\tilde{E}, \tilde{F})$ .

**3.3. A class of non-linear maps between Hilbert bundles.** Suppose now that  $\mathcal{V}$ ,  $\mathcal{W}$  are real Hilbert spaces,  $\mathcal{E}$ ,  $\mathcal{F}$  are Hilbert bundles over the compact basis  $B$ , and let  $\mu : \mathcal{E} \times \mathcal{V} \rightarrow \mathcal{F} \times \mathcal{W}$  be a continuous  $S^1$ -equivariant map over  $B$  which is fiberwise  $\mathcal{C}^\infty$ , and whose fiberwise derivatives are continuous on  $\mathcal{E} \times \mathcal{V}$ . We will also assume that the fiberwise differentials

$$d_y := d_{0_y} \mu_y = \mathcal{E}_y \times \mathcal{V} \longrightarrow \mathcal{F}_y \times \mathcal{W} , \quad y \in B$$

at the origins of the fibers  $\mathcal{E}_y \times \mathcal{V}$  are Fredholm. The linear operator  $d_y$  has the form  $d_y = (\delta_y, l_y)$ , where  $\delta_y : \mathcal{E}_y \rightarrow \mathcal{F}_y$  and  $l_y : \mathcal{V} \rightarrow \mathcal{W}$  are defined by the derivatives of the restrictions  $\mu|_{\mathcal{E}_y \times \{0^\mathcal{V}\}}$ ,  $\mu|_{\{0_y^\mathcal{E}\} \times \mathcal{V}}$ . Note that the continuous family  $\delta := (\delta_y)_{y \in B}$  of complex Fredholm operators defines an element  $\text{ind}(\delta) \in K(B)$ . Let  $d : \mathcal{E} \times \mathcal{V} \rightarrow \mathcal{F} \times \mathcal{W}$  the fiberwise linear map defined by the family of Fredholm operators  $(d_y)_{y \in B}$ . We suppose that  $\mu$  also has the properties

- P1:** (properness) There exist positive constants  $c$ ,  $C$  such that  $\|\mu(e, v)\| > c$  for all pairs  $(e, v) \in \mathcal{E} \times \mathcal{V}$  with  $\|(e, v)\| \geq C$ .
- P2:** (behavior near the  $S^1$ -fixed point set)
- (1)  $\mathcal{W}$  splits orthogonally as  $\mathcal{W} = H \oplus \mathcal{W}_0$ , where  $H$  is a finite dimensional subspace, and for every  $y \in B$  one has

$$\mu(0_y^\mathcal{E}, v) = h(y) + l(v) \quad \forall y \in B \quad \forall v \in \mathcal{V} .$$

where  $l : \mathcal{V} \xrightarrow{\cong} \mathcal{W}_0 \subset \mathcal{W}$  is a linear isometry.

In particular the operator  $l_y$  coincides with  $l$ , so is independent of  $y$ .

- (2) The map  $h$  is nowhere vanishing on  $B$ . Therefore there exists  $\varepsilon_0 > 0$  such that for every  $y \in B$  one has

$$\|h(y)\| = \|\text{p}_H(\mu(0_y^\mathcal{E}, v))\| \geq \varepsilon_0$$

- P3:** (linear+compactness) The difference  $k := \mu - d$  is compact, in the sense that for every  $R > 0$  the image  $k(D_R(\mathcal{E} \times \mathcal{V}))$  of the disk bundle  $(D_R(\mathcal{E} \times \mathcal{V}))$  is relatively compact in the total space  $\mathcal{F} \times \mathcal{W}$ .

Note that one has the identity

$$k(0_y^{\mathcal{E}}, v) = h(y) \in H, \quad \forall y \in B. \quad (15)$$

**3.4. The Seiberg-Witten map in dimension 4.** Let  $M$  be closed oriented 4-manifold, and let  $L$  be a Hermitian line bundle on  $M$ . We fix the following data:

- (1) A closed complement  $\mathcal{S}$  of the closed subspace  $iB_{\text{DR}}^1(M) = d(iA^0(M))$  of  $iA^1(M)$ .
- (2) A closed complement  $\mathcal{V}$  of the finite dimensional space

$$i\mathbb{H}^1 := \mathcal{S} \cap \ker(d : iA^1(M) \rightarrow iA^2(M)) \simeq iH^1(M, \mathbb{R})$$

in  $\mathcal{S}$

- (3) A complement  $i\mathbb{H}^2$  of  $d(iA^1(M))$  in  $\ker(d : iA^2(M) \rightarrow iA^3(M))$ . This complement will come with an isomorphism  $i\mathbb{H}^2 \simeq iH^2(M, \mathbb{R})$ .
- (4) An affine subspace  $\mathcal{A}$  of the space of connections  $\mathcal{A}(L)$  modeled after  $\mathcal{S}$ .

Therefore,  $\mathcal{A}$  is a slice to the orbits of the right action of the gauge group  $\mathcal{G}$  on the space of connections:

$$a \cdot g := a + 2g^{-1}dg$$

The quotient  $\bar{\mathcal{A}} := \mathcal{A}/\mathcal{V}$  is an affine space modeled after  $iH^1(M, \mathbb{R})$ . Consider the finite dimensional Lie group

$$G := \{u \in \mathcal{C}^\infty(M, S^1) \mid u^{-1}du \in \mathcal{S}\}.$$

One has an obvious short exact sequence

$$\{1\} \longrightarrow S^1 \longrightarrow G \xrightarrow{\lambda} 2\pi iH^1(M, \mathbb{Z}) \longrightarrow \{1\},$$

where  $\lambda$  is defined by  $u \mapsto [u^{-1}du]_{\text{DR}}$ . The choice of a point  $x_0 \in M$  defines a left splitting  $ev_{x_0} : G \rightarrow S^1$  whose kernel is isomorphic to  $2\pi iH^1(M, \mathbb{Z})$  and which will be denoted by  $G_{x_0}$ . In the affine space  $\mathcal{A}$  we have a natural  $i\mathbb{H}^1$ -invariant (hence  $G_{x_0}$ -invariant) subset  $\mathcal{A}_0$  defined by

$$\mathcal{A}_0 := \{a \in \mathcal{A} \mid F_a \in i\mathbb{H}^2\}.$$

The curvature  $F_{a_0}$  of a connection  $a_0 \in \mathcal{A}_0$  is independent of  $a_0$ , because it coincides with the representative in  $i\mathbb{H}^2$  of the de Rham class  $-2\pi ic_1^{\text{DR}}(L)$ ; this 2-form will be denoted by  $F_0$ . Note that  $\mathcal{A}_0$  is a  $G_{x_0}$ -invariant complete system of representatives for the quotient  $\bar{\mathcal{A}} = \mathcal{A}/\mathcal{V}$ . The space  $\mathcal{A}/G_{x_0}$  can be regarded as an affine bundle over the torus

$$\text{Pic}(L) := \bar{\mathcal{A}}/G_{x_0},$$

which is naturally a  $iH^1(X, \mathbb{R})/4\pi iH^1(X, \mathbb{Z})$ -torsor. The fibers of the affine bundle

$$\pi : \mathcal{A}/G_{x_0} \longrightarrow \text{Pic}(L)$$

are affine  $\mathcal{V}$ -spaces. Since the quotient  $\mathcal{A}_0/G_{x_0}$  is a section of this affine bundle, we can regard it as a  $\mathcal{V}$ -vector bundle over  $\text{Pic}(L)$  with  $\mathcal{A}_0/G_{x_0}$  as zero section. This vector bundle is actually trivial: indeed, the map  $(a_0, v) \mapsto a_0 + v \in \mathcal{A}$  is  $G_{x_0}$ -equivariant, and it descends to a trivialization  $\text{Pic}(L) \times \mathcal{V} \rightarrow \mathcal{A}/G_{x_0}$ .

**Remark 3.6.** *Choosing a Riemannian metric  $g$  gives canonical choices for the three objects  $S$ ,  $T$ ,  $i\mathbb{H}^2$  above, namely*

$$S = \ker(d^* : iA^1(M) \longrightarrow iA^0(M)), \quad \mathcal{V} := d^*(iA^2(M)), \quad i\mathbb{H}^2 = i\mathbb{H}_g^2,$$

where the subscript  $g$  on the right denotes the respective  $g$ -harmonic space. With these choices,  $\mathcal{A}_0$  is just the set of  $g$ -Yang-Mills connections in the slice  $\mathcal{A}$ .

Let  $g$  be a Riemannian metric on  $M$ , and let  $\tau : Q \rightarrow P_g$  be a  $Spin^c$ -structure on  $M$ . Denote by  $\Sigma^\pm$ ,  $\Sigma := \Sigma^- \oplus \Sigma^+$  the spinor bundles of  $\tau$ ,  $L = \det(\Sigma^\pm)$  the determinant line bundle, and  $\gamma : \Lambda^1 \rightarrow \text{End}_0(\Sigma)$  the Clifford map [OT].

The gauge group  $\mathcal{G}$  and its subgroup  $G_{x_0}$  act from the left on the vector spaces of sections  $A^0(\Sigma^\pm)$  by the formula

$$(g, \Psi) \mapsto g^{-1}\Psi .$$

Since  $G_{x_0}$  acts freely on the affine quotient space  $\bar{\mathcal{A}}$  we get two flat vector bundles  $\bar{\mathcal{A}} \times_{G_{x_0}} A^0(\Sigma^\pm)$  over  $\text{Pic}(L)$  with standard fibers  $A^0(\Sigma^\pm)$ . In order to use our general formalism we make the following definitions:

$$B := \text{Pic}(L) , \quad \mathcal{E} := \bar{\mathcal{A}} \times_{G_{x_0}} A^0(\Sigma^+) , \quad \mathcal{F} := \bar{\mathcal{A}} \times_{G_{x_0}} A^0(\Sigma^-) , \quad \mathcal{W} := iA_+^2(M) .$$

Let  $\kappa : B \rightarrow i\mathbb{H}_g^+$  be a smooth map. The  $\kappa$ -twisted Seiberg-Witten map is the map from  $A^0(\Sigma^+) \times \mathcal{A}$  to  $A^0(\Sigma^-) \times iA_+^2$  given by

$$(\Psi, a) \mapsto (\not{D}_a \Psi, (F_a - F_0 + \kappa(\pi(a)))^+ - \gamma^{-1}((\Psi \bar{\Psi})_0)) .$$

Via the identification  $B \times \mathcal{V} = \mathcal{A}/G_{x_0}$  this map descends to an  $S^1$ -equivariant map

$$sw_\kappa : \mathcal{E} \times \mathcal{V} \longrightarrow \mathcal{F} \times \mathcal{W} .$$

the restriction of  $sw_\kappa$  to the fiber over  $y = [a_0] \in B$  is given by the formula

$$sw_\kappa(\Psi, v) = \left( \not{D}_{a_0} \Psi + \frac{1}{2} \gamma(v) \Psi , d^+ v + \kappa(y) - \gamma^{-1}((\Psi \bar{\Psi})_0) \right) .$$

The linearization of this map at the zero section in the bundle  $\mathcal{E} \times \mathcal{V}$  over  $B$  is a fiberwise linear bundle map given by

$$d(\Psi, v) = (\not{D}_{a_0} \Psi, d^+ v) .$$

Hence  $sw_\beta$  decomposes as

$$sw_\kappa = d + c_\kappa ,$$

where  $c_\kappa$  is the sum of a quadratic map  $c$  and the fiberwise constant map defined by  $\kappa$ . Denote by  $w_\tau$  the expected dimension of the Seiberg-Witten moduli space corresponding to  $\tau$ :

$$w_\tau := \frac{1}{4}(c_1(L)^2 - 3\sigma(M) - 2e(M))$$

We define Sobolev  $L_k^2$ -completions of the spaces  $\mathcal{V}$ ,  $\mathcal{W}$  in the usual way. The construction of Sobolev norms on the bundles  $\mathcal{E}$ ,  $\mathcal{F}$  is more delicate, because these bundles as quotients with respect with respect to group  $\mathcal{G}_0$ , which does not operate by  $L_k^2$ -isometries<sup>2</sup>. For a point  $y = [a_0] \in B$  (with  $a_0 \in \mathcal{A}_0$ ) one identifies the fiber  $\mathcal{E}_y$ ,  $\mathcal{F}_y$  with  $\{a_0\} \times A^0(\Sigma^\pm)$  and uses the covariant derivatives associated with  $\nabla_{a_0}$  to define the  $L_k^2$ -norm on  $\mathcal{E}_y$ . A gauge transformation  $g \in G_0$  will define an isometry  $\{a_0\} \times A^0(\Sigma^+) \rightarrow \{a_0 \cdot g\} \times A^0(\Sigma^\pm)$ , so in this way one obtains a well defined Sobolev norm on the fiber  $\mathcal{E}_y$ .

**Lemma 3.7.** *With respect to suitable Sobolev completions, the following holds:*

- (1)  $sw_\kappa$  is smooth.
- (2) The fiberwise linear map  $d$  is fiberwise Fredholm of index  $w_\tau - b_1 + 1$ , and  $c_\beta$  is a compact map.

<sup>2</sup>We are grateful to Markus Bader for pointing us out this subtlety.



(3) *There exists positive constants  $c, C$  such that*

$$\|(\Psi, v)\| \geq C \Rightarrow \|sw_\kappa(\Psi, v)\| > c .$$

(4) *The map  $c_\kappa = sw_\kappa - d$  is compact.*

Therefore the Seiberg-Witten map  $sw_\kappa$  satisfies always the properties  $\mathcal{P}1$ ,  $\mathcal{P}2$  (1) and  $\mathcal{P}3$  in section 3.3. It also satisfies  $\mathcal{P}2$  (2) when  $\kappa : B \rightarrow i\mathbb{H}_g^+ \setminus \{0\}$ .

The first and the third statements in the lemma are easy to see. The crucial properness assertion (2) is stated in [Fu1], [Fu2]. A proof of the analogue statement for another version of the Seiberg-Witten map can be found in [BF]). A detailed proof for our version, and an analogue properness property in a different gauge theoretic context can be found in [B]. Similar methods can be also used to treat the 3-dimensional Casson-Seiberg-Witten theory.

**3.5. Finite dimensional approximation.** We will need the following simple geometric construction. Let  $\mathcal{A}$  be a (real or complex) Hilbert space, and  $A \subset \mathcal{A}$  a finite dimensional subspace. Following [BF] we introduce, for every  $\varepsilon > 0$  the retraction

$$\rho_{\varepsilon, A} : \mathcal{A}^+ \setminus S_\varepsilon(A^\perp) \rightarrow A^+$$

in the following way. For every  $a \in A \setminus \{0\}$ , put

$$s_{\varepsilon, a} := \frac{\|a\|^2 - \varepsilon^2}{2\|a\|^2} , \quad c_{\varepsilon, a} = s_{\varepsilon, a}a , \quad r_{\varepsilon, a} := \frac{\|a\|^2 + \varepsilon^2}{2\|a\|} .$$

Let  $S_{\varepsilon, a} \subset \mathbb{R}a + A^\perp$  be the hypersphere of  $\mathbb{R}a + A^\perp$  defined by the equation

$$\|b - c_{\varepsilon, a}\|^2 + \|a'\|^2 = r_{\varepsilon, a}^2 .$$

The hypersphere  $S_{\varepsilon, a}$  has the properties

$$a \in S_{\varepsilon, a} , \quad S_\varepsilon(A^\perp) \subset S_{\varepsilon, a} .$$

Consider also the spherical calotte:

$$C_{\varepsilon, a} := \{ta + a' \in S_{\varepsilon, a} \mid t > 0\} \subset S_{\varepsilon, a} .$$

Denote by  $C_{\varepsilon, \infty} \subset [A^\perp]^+$  the exterior of the sphere  $S_\varepsilon(A^\perp) \subset A^\perp$  (including  $\infty$ ), and by  $C_{\varepsilon, 0}$  its interior. Now note that

$$\mathcal{F}_{\varepsilon, A} := \{C_{\varepsilon, a} \mid a \in A^+\}$$

is a foliation of  $\mathcal{A}^+ \setminus S_\varepsilon(A^\perp)$  with closed leaves; the leaves are all diffeomorphic to the standard disk of  $A^\perp$ . The retraction  $\rho_{\varepsilon, A}$  assigns the point  $a \in A^+$  to any point of the leaf  $C_{\varepsilon, a} \subset \mathcal{A}^+$ . Note that for any  $z \in \mathcal{A}$  one has the implication

$$(z \in \mathcal{A}^+ \setminus S_\varepsilon(A^\perp), \|z\| \geq \varepsilon) \Rightarrow \|\rho_{\varepsilon, A}(z)\| \geq \|z\| \quad (16)$$

(equality is obtained when  $\|z\| = \varepsilon$  or  $z \in A$ ). A second important property of the retraction  $\rho_{\varepsilon, A}$  is

$$z \in \mathcal{A} \setminus A^\perp \Rightarrow \rho_{\varepsilon, A}(z) = \lambda_{\varepsilon, z} p_A(z) \text{ with } \lambda_{\varepsilon, z} \geq 1 . \quad (17)$$

Any  $\mathbb{R}$ -linear isometry  $u$  of  $\mathcal{A}$  which leaves invariant the subspace  $A$  will also leave invariant the foliation  $\mathcal{F}_{\varepsilon, A}$ . Therefore

**Remark 3.8.**  $\rho_{\varepsilon, A}$  is equivariant with respect to any  $\mathbb{R}$ -linear isometry of  $\mathcal{A}$  which leaves invariant the subspace  $A$ .

These retractions play a fundamental role in the following construction of finite dimensional approximations. This construction is a refinement of the one developed in [BF]. The main difference is that we have to work over a base  $B$ , and that we treat the real and complex summands separately.

Consider again an  $S^1$ -equivariant map  $\mu : \mathcal{E} \times \mathcal{V} \rightarrow \mathcal{F} \times \mathcal{W}$  over  $B$  satisfying the properties  $\mathcal{P}1$ ,  $\mathcal{P}2$ ,  $\mathcal{P}3$  in section 3.3. Recall from section 3.3 that we denoted by  $d$  the linearization of  $\mu$  at the 0-section and by  $\delta$  and  $l$  the complex and the real components of  $d$ . We may suppose that the  $\mathbb{R}$ -linear operator  $l$  induces an isometry  $\mathcal{V} \rightarrow \mathcal{W}_0$ . A finite rank subbundle  $F \subset \mathcal{F}$  will be called *admissible* if it is mapped surjectively onto the linear space defined by the family of cokernels  $(\text{coker}(\delta_y))_{y \in B}$ . A finite dimensional subspace  $W \subset \mathcal{W}$  will be called admissible if it contains  $H$ . A pair  $(F, W)$  will be called admissible if  $F$  and  $W$  are both admissible; in this case, for every  $y \in B$  the product  $F_y \times W$  is mapped surjectively onto  $\text{coker}(d_y)$ .

For every admissible pair  $\pi = (F, W)$  the preimage  $d^{-1}(F \times W)$  is a finite rank subbundle of  $\mathcal{E} \times \mathcal{V}$  which splits as

$$d^{-1}(F \times W) = \delta^{-1}(F) \times l^{-1}(W) .$$

We denote by  $W_0$  the orthogonal complement of  $H$  in  $W$ , and we put  $V := l^{-1}(W) = l^{-1}(W_0)$ ,  $E := \delta^{-1}(F) \subset \mathcal{E}$ . The pair  $(E, F)$  represents  $\text{ind}(\delta) \in K(B)$ . We get *topological orthogonal direct sum decompositions*

$$\mathcal{F} = F \oplus F^\perp , \quad \mathcal{E} = E \oplus E^\perp , \quad \mathcal{W} = W \oplus W^\perp = H \oplus W_0 \oplus W^\perp , \quad \mathcal{V} = V \oplus V^\perp .$$

The product  $F \times W$  is a finite dimensional Hilbert subbundle of  $\mathcal{F} \times \mathcal{W}$  whose orthogonal complement is  $F^\perp \times W^\perp$ . The retraction

$$\rho_{\varepsilon, F \times W} : [\mathcal{F} \times \mathcal{W}]_B^\perp \setminus S_\varepsilon(F^\perp \times W^\perp) \longrightarrow [F \times W]_B^\perp$$

is defined fiberwise. We will see that, for sufficiently small  $\varepsilon > 0$  and sufficiently large admissible pairs  $\pi = (F, W)$ , the image of the restriction  $\mu|_{E \times V}$  does not intersect  $S_\varepsilon(F^\perp \times W^\perp)$ , so that we can define

$$\mu_{\varepsilon, \pi} := \{\rho_{\varepsilon, F \times W} \circ \mu\}|_{E \times V} : E \times V \longrightarrow [F \times W]_B^\perp ,$$

obtaining a map between which belongs to the class studied in section 3.1. Such a map will be called a *finite dimensional approximation* of  $\mu$ . The result we need is very much similar to the first part of Lemma 2.3 in [BF]. We know that the preimage  $\mu^{-1}(D_C(\mathcal{F} \times \mathcal{W}))$  is contained in the disk bundle  $D_C(\mathcal{E} \times \mathcal{V}) \subset D_C(\mathcal{E}) \times D_C(\mathcal{V})$ . The image  $k(D_C(\mathcal{E}) \times D_C(\mathcal{V}))$  is relatively compact in the total space  $\mathcal{F} \times \mathcal{W}$ , because  $k$  is compact by property  $\mathcal{P}3$ . Now fix  $\eta > 0$  and let  $M_\eta$  be a finite subset of  $\mathcal{F} \times \mathcal{W}$  such that  $k(D_C(\mathcal{E}) \times D_C(\mathcal{V}))$  is contained in the union of the balls of radius  $\eta$  with centers in  $M_\eta$ .

A pair  $\pi := (F, W)$  will be called  $\eta$ -admissible if it is admissible and  $F \times W$  contains the finite set  $M_\eta$ . The set of  $\eta$ -admissible pairs is non-empty and cofinal in the set of pairs of finite dimensional subspaces  $(F, W)$ .

**Lemma 3.9.** (*Finite dimensional approximations*) *Let  $0 < \eta < \frac{\varepsilon}{4}$ . Then*

- (1) *For any  $\eta$ -admissible pair  $\pi = (F, W)$  one has*

$$\text{im}(\mu|_{E \times V}) \cap S_\varepsilon(F^\perp \times W^\perp) = \emptyset ,$$

*so the finite dimensional approximation*

$$\mu_{\varepsilon, \pi} := \{(\rho_{\varepsilon, F \times W}) \circ \mu\}|_{E \times V} : E \times V \longrightarrow [F \times W]_B^\perp$$

*is defined.*

- (2) The restriction  $\mu_{c,\pi}|_{D_C(F) \times D_C(V)}$  takes values in  $F \times W$ .
- (3) For any  $\eta$ -admissible pair  $\pi = (F, W)$  the finite dimensional approximation  $\mu_{c,\pi}$  satisfies the conditions **P1**, **P2** (see section 3.1) with the same constants  $C, c, \varepsilon_0$ , isometry  $l : \mathcal{V} \rightarrow \mathcal{W}_0 \subset \mathcal{W}$  and map  $h : B \rightarrow H$  as  $\mu$ .

**Proof:** 1. If the intersection  $\text{im}(\mu|_{E \times V}) \cap S_c(F^\perp \oplus W^\perp)$  was not empty, there would exist a point  $(e, v) \in E \times V$  such that  $\mu(e, v) \in S_c(F^\perp \times W^\perp)$ . Since  $S_c(F^\perp \times W^\perp) \subset D_c(\mathcal{F} \times \mathcal{W})$ , it follows  $(e, v) \in D_C(\mathcal{E}) \times D_C(\mathcal{V})$ . Therefore

$$\mu(e, v) = d(e, v) + k(e, v) \in F \times W_0 + k(D_C(\mathcal{E}) \times D_C(\mathcal{V})) .$$

But any element in the second set  $k(D_C(\mathcal{E}) \times D_C(\mathcal{V}))$  is  $\eta$ -close to an element in  $M_\eta \subset F \times W$ , so  $\mu(e, v)$  is  $\eta$ -close to  $F \times W$ . Since  $\eta < \frac{c}{4}$ , this contradicts  $\mu(e, v) \in S_c(F^\perp \oplus W^\perp)$ .

2. The same argument shows that  $\mu(D_C(E) \times D_C(V))$  does not intersect the complement of  $D_c(F^\perp \oplus W^\perp)$  in  $F^\perp \oplus W^\perp$ .

3. We have to check that, for an  $\eta$ -admissible pair  $\pi = (F, W)$ , the finite dimensional approximation  $\mu_{c,\pi}$  has the two properties **P1**, **P2** in section 3.1. For a point  $(e, v) \in E \times V$  with  $\|(e, v)\| \geq C$  it holds  $\|\mu(e, v)\| > c$  so, by (16), we have

$$\|\rho_{c,F \times W}(\mu(e, v))\| \geq \|\mu(e, v)\| > c . \quad (18)$$

On the other hand, for any  $y \in B, v \in V$  one has  $\mu(0_y^E, v) = h(y) + l(v) \in \{0_y^F\} \times W$ , hence

$$\mu_{c,\pi}(0_y^E, v) = \rho_{c,F \times W}(\mu(0_y^E, v)) = \mu(0_y^E, v) = h(y) + l(v) .$$

■

### 3.6. Compatibility properties.

**Lemma 3.10.** (Coherence Lemma) Let  $0 < \eta < \frac{c}{4}$ , let  $\pi = (F, W)$ ,  $\tilde{\pi} = (\tilde{F}, \tilde{W})$  be two  $\eta$ -admissible pairs with  $\pi \subset \tilde{\pi}$ , and let  $F', W'$  be the orthogonal complements of  $F, W$  in  $\tilde{F}, \tilde{W}$  respectively. The map

$$\mu_{c,\pi,\tilde{\pi}} := \iota \circ \left\{ [\mu_{c,\pi} \circ (\text{p}_E, \text{p}_V)] \wedge_B [(\text{p}_{F'}, \text{p}_{W'}) \circ (\delta, l)]_B^+ \right\} : \tilde{E} \times \tilde{V} \rightarrow \tilde{F} \times \tilde{W}$$

also satisfies properties **P1**, **P2** with constants  $C, \gamma$  (for a sufficiently small number  $0 < \gamma < c$ ),  $\varepsilon_0$ , and one has  $\{\mu_{c,\tilde{\pi}}\} = \{\mu_{c,\pi,\tilde{\pi}}\}$ .

**Proof:** The first statement follows from Proposition 3.5. We use the same method as in the proof of Lemma 2.3 in [BF] to construct a homotopy between the restriction of the two maps to the product  $D_C(\tilde{E}) \times D_C(V)$  and we will apply the homotopy invariance property of our invariant (see Proposition 3.2). The main difference compared to [BF] is that we have to control the restriction to the  $S^1$ -fixed point set, but we do not need an extension of the homotopy to the whole  $\tilde{E} \times \tilde{V}$ . For completeness we include detailed arguments adapted to our situation.

**Proof:** Denote by  $E', V'$  the orthogonal complements of  $E, V$  in  $\tilde{E}, \tilde{V}$ . We define the map

$$H : [0, 4] \times [D_C(\tilde{E}) \times D_C(\tilde{V})] \rightarrow [\mathcal{F} \times \mathcal{W}] \setminus \left[ \tilde{F}^\perp \times \tilde{W}^\perp \setminus \mathring{D}_c(\tilde{F}^\perp \times \tilde{W}^\perp) \right] \quad (19)$$

by the formula <sup>3</sup>

$$H_t = \begin{cases} d + [(1-t) \text{id}_{\mathcal{F} \times \mathcal{W}} + t \text{p}_{F \times W}] \circ k & \text{for } 0 \leq t \leq 1, \\ d + \text{p}_{F \times W} \circ k \circ [(2-t) \text{id}_{\tilde{E} \times \tilde{V}} + (t-1) \text{p}_{E \times V}] & \text{for } 1 \leq t \leq 2, \\ \text{p}_{F \times W} \circ k \circ \text{p}_{E \times V} + [d - (t-2) \text{p}_{F \times W} \circ d \circ \text{p}_{E' \times V'}] & \text{for } 2 \leq t \leq 3, \\ \text{p}_{F' \times W'} \circ d + [(4-t) \text{p}_{F \times W} + (t-3) \rho_{c, F \times W}] \circ \mu \circ \text{p}_{E \times V} & \text{for } 3 \leq t \leq 4. \end{cases}$$

**Claim:**  $H$  is a well defined, continuous,  $S^1$ -equivariant map over  $B$ .

This follows from:

a) For a point  $(t, \tilde{e}, \tilde{v}) \in [0, 4] \times D_C(\tilde{E}) \times D_C(\tilde{V})$ , the term  $\rho_{c, F \times W}(\mu(\text{p}_{E \times V}(\tilde{e}, \tilde{v})))$  is finite, so the convex combination in the fourth branch is defined and finite.

Indeed, recall that the retraction  $\rho_{c, F \times W}$  is finite on the complement of the leaf  $[F^\perp \times W^\perp] \setminus D_c(F^\perp \times W^\perp)$ . Therefore it suffices to note that  $k(D_C(\tilde{E}) \times D_C(\tilde{V}))$  is  $\eta$ -close to  $F \times W$  and  $d(E \times V) \subset F \times W$ , so the point  $\mu(\text{p}_{E \times V}(\tilde{e}, \tilde{v}))$  is  $\eta$ -close to  $F \times W$  for  $(\tilde{e}, \tilde{v}) \in D_C(\tilde{E}) \times D_C(\tilde{V})$ . Therefore

$$\mu(\text{p}_{E \times V}(\tilde{e}, \tilde{v})) \notin [F^\perp \times W^\perp] \setminus \mathring{D}_c(F^\perp \times W^\perp).$$

b) The formulae given for the four components of  $H$  agree on the intersections of their domains.

c)  $H$  takes values in  $[\mathcal{F} \times \mathcal{W}] \setminus [\tilde{F}^\perp \times \tilde{W}^\perp \setminus \mathring{D}_c(\tilde{F}^\perp \times \tilde{W}^\perp)]$ .

Indeed, for  $(t, \tilde{e}, \tilde{v}) \in [0, 4] \times D_C(\tilde{E}) \times D_C(\tilde{V})$  we see as in the proof of a) that the right hand term of  $H_t$  must be  $\eta$ -close to  $\tilde{F} \times \tilde{W}$ , so  $H([0, 4] \times D_C(\tilde{E}) \times D_C(\tilde{V}))$  avoids  $[F^\perp \times W^\perp] \setminus \mathring{D}_c(F^\perp \times W^\perp)$ .

The map  $H$  has the following important properties:

- (1)  $H_0$  coincides with the restriction  $\mu|_{D_C(\tilde{E}) \times D_C(\tilde{V})}$ .
- (2)  $H_4$  coincides with the map  $\mu_{c, \pi, \tilde{\pi}}$  composed with the inclusion  $\tilde{F} \times \tilde{V} \hookrightarrow [\mathcal{F} \times \mathcal{W}]_B^+ \setminus S_c(\tilde{F}^\perp \times \tilde{W}^\perp)$ .
- (3) One has

$$H_t(0_{\tilde{y}}^{\tilde{E}}, \tilde{v}) = h(y) + l(\tilde{v}), \quad \forall t \in [0, 4] \quad \forall y \in B \quad \forall \tilde{v} \in D_C(\tilde{V}), \quad (20)$$

Formula (20) follows from (15) and the fact that  $l$  is an isometry, so it commutes with orthogonal projections.

- (4)  $H([0, 4] \times \partial(D_C(\tilde{E}) \times D_C(\tilde{V})) \cap [\tilde{F}^\perp \times \tilde{W}^\perp]) = \emptyset$ .

Indeed, for  $(\tilde{e}, \tilde{v}) \in \partial(D_C(\tilde{E}) \times D_C(\tilde{V}))$  one has  $\|H_0(\tilde{e}, \tilde{v})\| = \|\mu(\tilde{e}, \tilde{v})\| \geq c$ , whereas  $\|\mu(\tilde{e}, \tilde{v})\|$  is  $\eta$ -close to  $F \times W \subset \tilde{F} \times \tilde{W}$ . Moreover, for  $t \in [0, 1]$  one has  $\|H_t(\tilde{e}, \tilde{v}) - H_0(\tilde{e}, \tilde{v})\| = t\|(\text{p}_{F^\perp \times W^\perp} \circ k)(\tilde{e}, \tilde{v})\| \leq \eta$ . For  $t \geq 2$  we have

$$\text{p}_{F' \times W'} \circ h_t = \text{p}_{F' \times W'} \circ d,$$

so  $H_t(\tilde{e}, \tilde{v})$  can belong to  $\tilde{F}^\perp \times \tilde{W}^\perp$  only when  $\text{p}_{F' \times W'} \circ d(\tilde{e}, \tilde{v}) = 0$ , i.e. when  $(\tilde{e}, \tilde{v}) \in E \times V$ . For such a pair we have

$$H_t(\tilde{e}, \tilde{v}) = (d + \text{p}_{F \times W} \circ k)(\tilde{e}, \tilde{v}) = \mu(\tilde{e}, \tilde{v}) - (\text{p}_{F^\perp \times W^\perp} \circ k)(\tilde{e}, \tilde{v}) \quad \forall t \in [1, 3],$$

$$H_t(\tilde{e}, \tilde{v}) \in [\text{p}_{F \times W}(\mu(\tilde{e}, \tilde{v})), \rho_{c, F}(\mu(\tilde{e}, \tilde{v}))] \quad \forall t \in [3, 4],$$

<sup>3</sup>The third branch of the homotopy was omitted in [BF].

so  $H_t(\tilde{e}, \tilde{v})$  is a non-vanishing vector of  $F \times W$  (more precisely a positive multiple of  $\mathfrak{p}_{F \times W}(\mu(\tilde{e}, \tilde{v})) = \mu(\tilde{e}, \tilde{v}) - (\mathfrak{p}_{F^\perp \times W^\perp} \circ k)(\tilde{e}, \tilde{v})$ ) for any  $t \in [1, 4]$ .

These properties have the following important consequence:

**Remark:** *The composition  $\rho_{c, \bar{\pi}} \circ H$  is nowhere vanishing on the space*

$$[0, 4] \times \left\{ \partial \left[ D_C(\tilde{E}) \times D_C(\tilde{V}) \right] \cup \left[ 0^{\tilde{E}} \times \tilde{V} \right] \right\} .$$

This follows from the fact that the vanishing locus of the retraction  $\rho_{c, \bar{\pi}}$  is the leaf  $\mathring{D}_C(\tilde{F}^\perp \times \tilde{W}^\perp) \subset \tilde{F}^\perp \times \tilde{W}^\perp$ . On the other hand one has

$$\rho_{c, \bar{\pi}} \circ H_0 = \mu_{c, \bar{\pi}}|_{D_C(\tilde{E}) \times D_C(\tilde{V})} , \quad \rho_{c, \bar{\pi}} \circ H_4 = \mu_{c, \pi, \bar{\pi}}|_{D_C(\tilde{E}) \times D_C(\tilde{V})}$$

It suffices now to apply Proposition 3.2. ■

Using Proposition 3.5 and Lemma 3.10 we obtain

**Corollary 3.11.** *Let  $\mu : \mathcal{E} \times \mathcal{V} \rightarrow \mathcal{F} \times \mathcal{W}$  be an  $S^1$ -equivariant map over a compact base  $B$  satisfying  $\mathcal{P}1, \mathcal{P}2, \mathcal{P}3$ , and let  $0 < \eta < \frac{c}{4}$ . Fix an orientation  $\mathfrak{o}$  of the finite dimensional summand  $H$  of  $\mathcal{W}$ . The elements*

$$\{\mu_{c, \pi}\} \in S^1 \alpha_B^{b-1}(S(E)_{+B}, F_B^+)$$

*associated with  $\eta$ -admissible pairs  $(F, W)$  define a unique class  $\{\mu\} \in \alpha^{b-1}(\text{ind}(\delta))$  which depends only on the map  $\mu$  and the orientation  $\mathfrak{o}$ .*

In particular, using finite dimensional approximations associated with constants  $C' \geq C$  and  $0 < c' \leq c$  (and parameter  $0 < \eta < \frac{c'}{4}$ ), one will obtain the same class.

**Proposition 3.12.** *Suppose that the restriction  $\mu|_{D_C(\mathcal{E}) \times D_C(V)}$  is nowhere vanishing. Then  $\{\mu\} = 0$ .*

**Proof:** Since  $\mu|_{D_C(\mathcal{E}) \times D_C(V)}$  is nowhere vanishing, it is easy to see that there exists  $\gamma > 0$  such that  $\|\mu(e, v)\| > c'$  for every  $(e, v) \in D_C(\mathcal{E}) \times D_C(V)$ . Indeed, if not there would exist a sequence  $(e_n, v_n) \in D_C(\mathcal{E}) \times D_C(V)$  such that  $\|\mu(e_n, v_n)\| \rightarrow 0$ . Let  $K \subset \mathcal{F} \times \mathcal{W}$  be a compact subspace which contains  $k(D_C(\mathcal{E}) \times D_C(V))$ . Since  $d = (\delta, l)$  is a continuous family of Fredholm operators, it follows that  $d^{-1}(K) \cap [D_C(\mathcal{E}) \times D_C(V)]$  is compact. Therefore  $(e_n, v_n)_n$  admits a subsequence which converges in this intersection. The limit will be a vanishing point of  $\mu$ , which contradicts the assumption.

Use now the constant  $c' := \min(\gamma, c)$  (instead of  $c$ ) in the construction of the finite dimensional approximations of  $\mu$ . The obtained maps  $\mu_{c', \pi}$  will be nowhere vanishing on  $D_C(E) \times D_C(V)$ . It suffices to apply the vanishing property Proposition 3.1 proved in the finite dimensional case. ■

## 4. FUNDAMENTAL PROPERTIES OF THE COHOMOTOPY INVARIANTS

### 4.1. The Hurewicz image of the cohomotopy invariant.

4.1.1. *The relative Hurewicz morphism.* Let  $B$  be a compact space,  $E, F$  Hermitian bundles of ranks  $e, f$  over  $B$ . Let  $k$  be an integer and  $u \in \alpha^k(S(E)_{+B}, F_B^+)$  a stable class. Suppose for simplicity  $k \geq 0$ . Consider a representative

$$\varphi : S(E)_{+B} \wedge_B \xi_B^+ \rightarrow F_B^+ \wedge_B [\mathbb{R}^k]_B^+ \wedge \xi_B^+$$

of this stable class, where  $\xi = \eta \oplus \xi_0$  is the direct sum of a complex vector bundle  $\eta$  and a real vector bundle  $\xi_0$ . The space  $S(E)_{+B} \wedge_B \xi_B^+$  can be identified

with the fiberwise quotient  $\{S(E) \times_B \xi_B^+\}/_B \{S(E) \times_B \infty_\xi\}$ . Composing  $\varphi$  with the canonical projection one obtains a map of pairs over  $B$

$$\tilde{\varphi} : (S(E) \times_B \xi_B^+, S(E) \times_B \infty_\xi) \rightarrow ([F \oplus \mathbb{R}^k \oplus \xi]_B^+, \infty_{F \oplus \mathbb{R}^k \oplus \xi}) .$$

Consider the projection  $\pi : \mathbb{P}(E) \rightarrow B$  and the following bundles over  $\mathbb{P}(E)$ :

$$\tilde{F} := \pi^*(F)(1) , \quad \tilde{\xi} := \pi^*(\xi)(1) \oplus \pi^*(\xi_0) .$$

The map  $\tilde{\varphi}$  descends to a morphism of pointed sphere bundles over  $\mathbb{P}(E)$

$$\tilde{\varphi} : \tilde{\xi}_{\mathbb{P}(E)}^+ \longrightarrow [\tilde{F} \oplus \mathbb{R}^k \oplus \tilde{\xi}]_{\mathbb{P}(E)}^+ .$$

Denote by  $s$  the real rank of  $\xi$  and choose an orientation of the real bundle  $\xi_0$ . In this way all our bundles become oriented bundles. Denote by

$$t_{\tilde{\xi}} \in H^s(\tilde{\xi}_{\mathbb{P}(E)}^+, \infty_{\tilde{\xi}}) , \quad t_{\tilde{F} \oplus \mathbb{R}^k \oplus \tilde{\xi}} \in H^{2f+k+s}([\tilde{F} \oplus \mathbb{R}^k \oplus \tilde{\xi}]_{\mathbb{P}(E)}^+, \infty_{\tilde{F} \oplus \mathbb{R}^k \oplus \tilde{\xi}})$$

the Thom classes of the oriented bundles  $\tilde{\xi}$ ,  $\tilde{F} \oplus \mathbb{R}^k \oplus \tilde{\xi}$ . The formula

$$\tilde{\varphi}^*(t_{\tilde{F} \oplus \mathbb{R}^k \oplus \tilde{\xi}}) = p_{\mathbb{P}(E)}^*(h_{\tilde{\varphi}}) \cup t_{\tilde{\xi}}$$

defines a cohomology class  $h_{\tilde{\varphi}} \in H^{2f+k}(\mathbb{P}(E), \mathbb{Z})$  which is independent of the chosen orientation of  $\xi_0$  and of the representative  $\varphi$  of the stable class  $u$ . For  $k \leq 0$  one uses a similar construction, but uses a  $[\mathbb{R}^{-k}]_B^+$  factor on the left side.

The assignment  $u = [\varphi] \mapsto h_{\tilde{\varphi}}$  defines a morphism

$$\chi : \alpha^k(S(E)_{+B}, F_B^+) \rightarrow H^{2f+k}(\mathbb{P}(E), \mathbb{Z}) ,$$

which we call the *relative Hurewicz morphism* over  $B$ .

Denote by  $q : \tilde{\xi} \rightarrow \mathbb{P}(E)$  the bundle projection, and by  $\tilde{\varphi}$  the section defined by  $\tilde{\varphi}$  in the pull-back  $q^*([\tilde{F} \oplus \mathbb{R}^k \oplus \tilde{\xi}]_{\tilde{\xi}}^+)$  over  $\tilde{\xi}$ . The vanishing locus  $Z(\tilde{\varphi})$  of this section is compact. Since  $\tilde{\varphi}$  is a section with compact vanishing locus, one can define its *localized Euler class* class  $[\tilde{\varphi}] \in H_{d+2e-2-2f-k}(\tilde{\xi}, \mathbb{Z})$ , which coincides with the fundamental class  $[Z(\tilde{\varphi})]$  of the compact oriented submanifold  $[Z(\tilde{\varphi})]$  when this section is smooth and transversal to the zero section [Br].

**Remark 4.1.** (*The geometric interpretation of the Hurewicz morphism*) Suppose that  $B$  is an oriented  $d$ -dimensional compact manifold. Then

$$PD_{\mathbb{P}(E)}(\chi(u)) = [\iota_*]^{-1}([\tilde{\varphi}] ,$$

where

$$\iota_* : H_{d+2e-2-2f-k}(\mathbb{P}(E), \mathbb{Z}) \rightarrow H_{d+2e-2-2f-k}(\tilde{\xi}, \mathbb{Z}) .$$

is the isomorphism induced by the zero section of  $\tilde{\xi}$ . If  $\tilde{\varphi}$  is smooth and transversal to the zero section, then

$$PD_{\mathbb{P}(E)}(\chi(u)) = [\iota_*]^{-1}([Z(\tilde{\varphi})]) .$$

**Proof:** The localized Euler class  $[\tilde{\varphi}] \in H_{d+2e-2-2f-k}(\tilde{\xi}, \mathbb{Z})$  is defined as the cap product  $\tilde{\varphi}^*(t_{q^*(\tilde{F} \oplus \mathbb{R}^k \oplus \tilde{\xi})}) \cap [\tilde{\xi}]$ , where  $[\tilde{\xi}]$  stands for the fundamental class of  $\tilde{\xi}$  in cohomology with compact supports [Br]. We get

$$\begin{aligned} [\tilde{\varphi}] &= \tilde{\varphi}^*(t_{\tilde{F} \oplus \mathbb{R}^k \oplus \tilde{\xi}}) \cap [\tilde{\xi}] = [p_{\mathbb{P}(E)}^*(\chi(u)) \cup t_{\tilde{\xi}}] \cap [\tilde{\xi}] = \\ &= p_{\mathbb{P}(E)}^*(\chi(u)) \cap \iota_*([\mathbb{P}(E)]) = \iota_*(\chi(u) \cap [\mathbb{P}(E)]) = \iota_*(PD_{\mathbb{P}(E)}(\chi(u))) . \end{aligned}$$

■

Let  $\nu = (i, E_1) : E \rightarrow E'$  be a morphism in the category  $\mathcal{U}_B$  of complex vector bundles over  $B$  (see section 2.3). Such a morphism induces an isomorphism  $E' \simeq E \oplus E_1$ . The complement  $\mathbb{P}(E') \setminus \mathbb{P}(E_1)$  can be identified with the total space of the complex vector bundle  $\pi^*(E_1)(1) \rightarrow \mathbb{P}(E)$ . Multiplication with the Thom class  $t_N$  defines a morphism

$$\begin{aligned} H^*(\mathbb{P}(E), \mathbb{Z}) &\longrightarrow H^{*+2e_1}(\pi^*(E_1)(1)_{\mathbb{P}(E)}^\dagger, \infty_{\pi^*(E_1)(1)}) \cong \\ &\cong H^{*+2e_1}(\mathbb{P}(E'), \mathbb{P}(E_1)) \longrightarrow H^{*+2e_1}(\mathbb{P}(E'), \mathbb{Z}) , \end{aligned}$$

which will be denoted by  $a_\nu$ .

Fix now an element  $x \in K(B)$ . A morphism  $\tau = (i, j; E_1, F_1, l) : (E, F) \rightarrow (E', F')$  in the category  $\mathcal{T}(x)$  defines morphisms

$$\begin{aligned} a_{(i, E_1)} : H^{2f+k}(\mathbb{P}(E), \mathbb{Z}) &\rightarrow H^{2f'+k}(\mathbb{P}(E'), \mathbb{Z}) , \\ \mathbb{P}(i)_* : H_k(\mathbb{P}(E), \mathbb{Z}) &\rightarrow H_k(\mathbb{P}(E'), \mathbb{Z}) . \end{aligned}$$

For an integer  $k \in \mathbb{Z}$  we define

$$H^k(x, \mathbb{Z}) := \varinjlim_{(E, F) \in x} H^{2f+k}(\mathbb{P}(E), \mathbb{Z}) , \quad H_k(x, \mathbb{Z}) := \varinjlim_{(E, F) \in x} H_k(\mathbb{P}(E), \mathbb{Z}) .$$

Using the same methods as in sections 2.1, 2.3 (stabilizing first with respect to trivial bundle enlargements) we see that these inductive limits exist in  $\mathcal{A}b$ .

**Remark 4.2.** (1) One has  $H_*(x, \mathbb{Z}) = H_*(B, \mathbb{Z}) \otimes \mathbb{Z}[t]$ .  
(2) For a compact  $n$ -dimensional CW complex  $B$  one has isomorphisms

$$H^k(x, \mathbb{Z}) \simeq \bigoplus_{\substack{\alpha - k \in 2\mathbb{Z} \\ \max(0, k - 2\iota(x) + 2) \leq \alpha \leq n}} H^\alpha(B, \mathbb{Z}) ,$$

where  $\iota(x) \in \mathbb{Z}$  is the index of  $x$ . In particular, putting  $n(x) := 2\iota(x) - 2 + n$ , one has  $H^{n(x)}(x, \mathbb{Z}) = H^n(B, \mathbb{Z})$ .

The integer  $n(x) := 2\iota(x) - 2 + n$  will be called *the dimension of the formal projectivization of  $x$* .

**Remark 4.3.** Suppose that  $B$  is a compact connected oriented manifold. The system of Poincaré duality isomorphisms  $PD_{\mathbb{P}(E)}$  defines isomorphisms

$$PD_x : H^k(x, \mathbb{Z}) \xrightarrow{\simeq} H_{n(x)-k}(x) .$$

**Remark 4.4.** The system of Hurewicz morphisms

$$\chi : \alpha^k(S(E)_{+B}, F_B^+) \rightarrow H^{2f+k}(\mathbb{P}(E), \mathbb{Z})$$

defines a morphisms of graded groups  $\chi_x : \alpha^*(x) \rightarrow H^*(x, \mathbb{Z})$ . If  $B$  is a compact connected oriented manifold, one also gets a morphism  $PD_x \circ \chi_x : \alpha^*(x) \rightarrow H_*(x, \mathbb{Z})$ , which will be called *the homological Hurewicz morphism*.

The following important result has the following significance: for moduli problem with vanishing “expected dimension”, the cohomotopy invariant yields the same information as the classical (co)homological information.

**Proposition 4.5.** *Suppose that  $B$  is a finite CW complex of dimension  $n$ . Then the Hurewicz morphism*

$$\chi_x^{n(x)} : \alpha^{n(x)}(x) \longrightarrow H^{n(x)}(x, \mathbb{Z}) = H^n(B, \mathbb{Z}) .$$

*is an isomorphism.*

**Proof:** Suppose  $n(x) \geq 0$  for simplicity. Fix a stabilizing bundle  $\xi$ . Using the same method and the same notations as in section 4.1.1 we see that the set

$$S^1 \pi^0(S(E)_{+B} \wedge_B \xi_B^+, F_B^+ \wedge_B [\mathbb{R}^{n(x)}]_B^+ \wedge \xi_B^+)$$

can be identified with the set of pointed bundle maps

$$\bar{\varphi} : \tilde{\xi}_{\mathbb{P}(E)}^+ \longrightarrow [\tilde{F} \oplus \mathbb{R}^{n(x)} \oplus \tilde{\xi}]_{\mathbb{P}(E)}^+$$

over  $\mathbb{P}(E)$ . This set can be identified with  $H^{\dim_{\mathbb{R}}(\mathbb{P}(E))}(\mathbb{P}(E), \mathbb{Z}) = H^n(B, \mathbb{Z})$  by Proposition 5.15 via the map  $\bar{\varphi} \mapsto h_{\bar{\varphi}}$ . The obtained bijections

$$S^1 \pi^0(S(E)_{+B} \wedge_B \xi_B^+, F_B^+ \wedge_B [\mathbb{R}^{n(x)}]_B^+ \wedge \xi_B^+) \simeq H^n(B, \mathbb{Z})$$

are compatible with morphisms  $\xi \rightarrow \xi'$  in the category  $\mathcal{C}_B$  and with morphisms  $(E, F) \rightarrow (E', F')$  in the category  $\mathcal{T}(x)$ , so we get a bijection  $\alpha^{n(x)}(x) \rightarrow H^n(B, \mathbb{Z})$  which coincide with the Hurewicz map, by the definition of this map.  $\blacksquare$

**4.1.2. A comparison theorem.** The main result of this section states: the *full* Seiberg-Witten type invariant obtained using the moduli space of solutions associated with a map  $\mu$  satisfying properties  $\mathcal{P}1$ ,  $\mathcal{P}2$ ,  $\mathcal{P}3$  can be identified with the image of the cohomotopy invariant under the homological Hurewicz map.

We begin with the finite dimensional case. Let  $B$  be a compact oriented manifold,  $p : E \rightarrow B$ ,  $q : F \rightarrow B$  Hermitian bundles over  $B$ , let  $V, W$  be Euclidean spaces, and let  $\mu : E \times V \rightarrow [F \times W]_B^+$  be an  $S^1$ -equivariant map over  $B$  satisfying properties **P1**, **P2** in section 3.1. The invariant  $\{\mu\} \in \alpha^{b-1}(S(E)_{+B}, F_B^+)$  is defined by a map of pairs

$$(S(E) \times D_R(\mathbb{R} \oplus V), S(E) \times S_R(\mathbb{R} \oplus V)) \rightarrow ([F \times W]_B^+, [F \times W]_B^+ \setminus \mathring{D}_\varepsilon(F \times W))$$

induced by the restriction  $\mu_{R,\varepsilon} : D_R(E) \times D_R(V) \rightarrow (F \times W)_B^+$  of  $\mu$  to a sufficiently large cylinder  $D_R(E) \times D_R(V)$ . The vanishing locus of  $\mu$  (regarded as section in the bundle  $(p^*(F) \times V) \times W \rightarrow E \times V$ ) is an  $S^1$ -invariant compact space contained in the open subspace  $\mathring{D}_R(E) \times \mathring{D}_R(V) \setminus [0^E \times_B D_R(V)]$  of the cylinder. Its  $S^1$ -quotient can be identified with the vanishing locus of the section  $\mathring{\mu}_{R,\varepsilon}$  induced by  $\mu_{R,\varepsilon}$  on the  $S^1$ -quotient  $\mathbb{P}(E) \times \mathring{D}_R(\mathbb{R} \oplus V)$  of  $S(E) \times D_R(\mathbb{R} \oplus V)$ . Using Remark 4.1 one obtains

**Corollary 4.6.** *Suppose that  $B$  is a compact oriented manifold. Via the isomorphism  $H_*(\mathbb{P}(E) \times \mathring{D}_R(\mathbb{R} \oplus V)) \simeq H_*(\mathbb{P}(E))$  the Poincaré dual  $PD_{\mathbb{P}(E)}(\chi(\{\mu\}))$  coincides with the virtual fundamental class associated with the section  $\mathring{\mu}_{R,\varepsilon}$ . If this section is smooth and transversal to the zero section, then  $PD_{\mathbb{P}(E)}(\chi(\{\mu\}))$  can be identified with the fundamental class of the vanishing locus  $Z(\mathring{\mu}_{R,\varepsilon}) \subset \mathbb{P}(E) \times \mathring{D}_R(\mathbb{R} \oplus V)$ .*

The vanishing locus  $M := Z(\mathring{\mu}_{R,\varepsilon}) \simeq Z(\mu)/S^1$  will be called the “moduli space” associated with the map  $\mu$ .



Let  $p : \mathcal{E} \rightarrow B$ ,  $q : \mathcal{F} \rightarrow B$  complex Hilbert bundles over  $B$ , let  $\mathcal{V}$ ,  $\mathcal{W}$  be real Hilbert spaces, and let  $\mu : \mathcal{E} \times \mathcal{V} \rightarrow \mathcal{F} \times \mathcal{W}$  be an  $S^1$ -equivariant map over  $B$  satisfying properties  $\mathcal{P}1$ ,  $\mathcal{P}2$ ,  $\mathcal{P}3$  in section 3.3. Denote by  $\pi : \mathbb{P}(\mathcal{E}) \rightarrow B$  the natural projection. The map  $\mu_{R,\varepsilon}$  descends to a smooth section  $\mathring{\mu}_{R,\varepsilon}$  in the bundle

$$\pi^*(\mathcal{F})(1) \times D_R(\mathbb{R} \oplus \mathcal{V}) \times \mathcal{W} \rightarrow \mathbb{P}(\mathcal{E}) \times \mathring{D}_R(\mathbb{R} \oplus \mathcal{V}) ,$$

and again one can identify the moduli space  $\mathcal{M} := Z(\mu)/S^1$  of  $\mu$  with the vanishing locus  $Z(\mathring{\mu}_{R,\varepsilon})$  of this section. Using the same argument as in the proof of Proposition 3.12, we see that the moduli space  $\mathcal{M}$  is compact. Suppose now that

$\mathcal{P}_4$ :  $B$  is a smooth, compact, oriented connected manifold,  $\mu$  is smooth and the fiberwise differential of  $k := \mu - d$  at any point is a compact operator.

This condition is always satisfied in practical gauge theoretical situations; indeed, the map  $k$  is usually given by the composition of a smooth map  $\mathcal{E} \times \mathcal{V} \rightarrow \mathcal{F}_1 \times \mathcal{W}_1$  with a map  $\mathcal{F}_1 \times \mathcal{W}_1 \rightarrow \mathcal{F} \times \mathcal{W}$  over  $B$  given by a smooth family of compact operators. The condition  $\mathcal{P}_4$  implies that  $\mathring{\mu}_{R,\varepsilon}$  is a smooth Fredholm section on the Banach manifold  $\mathbb{P}(\mathcal{E}) \times \mathring{D}_R(\mathbb{R} \oplus \mathcal{V})$ . In order to give sense to the virtual fundamental class of the moduli space  $\mathcal{M}$  we have to trivialize the determinant line bundle  $\det(\text{index}(D\mathring{\mu}_{R,\varepsilon}))$  over  $\mathcal{M}$ . Equivalently, it suffices to trivialize the line bundle  $\det(\text{index}(D\mu))$  over  $Z(\mu)$ . In these formulae the symbol  $D$  stands for the family of intrinsic derivatives of a section at its vanishing points, and  $\mu$  is regarded as a section in the bundle  $[p^*(\mathcal{F}) \times \mathcal{V}] \times \mathcal{W} \rightarrow \mathcal{E} \times \mathcal{V}$ . For a point  $(e, v) \in Z(\mu)$  with  $p(e) = y$  one has a natural identification

$$\det(\text{index}(D_{(e,v)}\mu)) = \Lambda^n(T_y(B)) \otimes \det(\text{index}(d_{(e,v)}\mu|_{\mathcal{E}_y \times \mathcal{V}})) ,$$

where  $n := \dim(B)$  and  $\mu|_{\mathcal{E}_y \times \mathcal{V}} : \mathcal{E}_y \times \mathcal{V} \rightarrow \mathcal{F}_y \times \mathcal{W}$  is the restriction of  $\mu$  to the fiber over  $y$ . By the condition  $\mathcal{P}_4$ , the differential of this restriction congruent with the operator  $d_y = (\delta_y, l)$  modulo a compact operator. Therefore (since the family  $\delta$  has a canonical complex orientation, and  $B$  is oriented) one obtains a trivialization of  $\det(\text{index}(D\mu))$  for every orientation  $\mathfrak{o}$  of  $\text{coker}(l) = H$ . This is precisely the orientation parameter involved in the definition of the cohomotopy invariant  $\{\mu\}$ . Fix such an orientation  $\mathfrak{o}$ . Using the results in [Br], we obtain a virtual fundamental class in Čech homology  $[\mathcal{M}]^{\text{vir}} \in \check{H}_w(\mathcal{M}, \mathbb{Z})$ , where  $w := n + 2\iota(x) - b - 1 = n(x) - (b - 1)$  is the expected dimension of our moduli problem (the index of the section  $\mathring{\mu}_{R,\varepsilon}$ ).

**Definition 4.7.** *The full gauge theoretical invariant of  $\mu$  is the image  $\{\mu\}_{GT}$  of the class  $[\mathcal{M}]^{\text{vir}}$  in the group*

$$H_w(\mathbb{P}(\mathcal{E}) \times \mathring{D}_R(\mathbb{R} \oplus \mathcal{V}), \mathbb{Z}) = H_w(\mathbb{P}(\mathcal{E}), \mathbb{Z}) = \bigoplus_{0 \leq 2i \leq w} H_{w-2i}(B, \mathbb{Z}) \otimes t^i = H_w(x, \mathbb{Z}) .$$

**Theorem 4.8.** *Suppose that conditions  $\mathcal{P}1 - \mathcal{P}4$  hold. Then*

$$\{\mu\}_{GT} = PD_x \circ \chi_x(\{\mu\}) .$$

**Proof:** Choose as in section 3.5 a finite dimensional approximation  $\mu_{c,\pi}$  of  $\mu$  associated with an  $\eta$ -admissible pair  $(F, W)$ . Define  $\mu_{c,\pi,\infty} : D_C(\mathcal{E}) \times D_C(\mathcal{V}) \rightarrow \mathcal{F} \times \mathcal{W}$  by

$$\mu_{c,\pi,\infty}(e, v) = \mu_{c,\pi}(\text{p}_E(e), \text{p}_V(v)) + \text{p}_{F^\perp \times W^\perp} \circ d \circ \text{p}_{E^\perp \times V^\perp} .$$

This map takes finite values by Lemma 3.9. We claim that there exists a smooth homotopy

$$\mathcal{H} : [0, 4] \times D_C(\mathcal{E}) \times D_C(\mathcal{V}) \rightarrow \mathcal{F} \times \mathcal{W}$$

between  $\mu|_{D_C(\mathcal{E}) \times D_C(\mathcal{V})}$  and  $\mu_{c,\pi,\infty}$  and  $\mu|_{D_C(\mathcal{E}) \times D_C(\mathcal{V})}$  in the space of  $S^1$ -equivariant Fredholm maps over  $B$ , such that for  $0 \leq t \leq 4$  the map  $\mathcal{H}_t$  has no vanishing point in  $\partial[D_C(\mathcal{E}) \times D_C(\mathcal{V})] \cup 0^\mathcal{E} \times D_C(\mathcal{V})$ . To obtain such a homotopy it suffices to replace  $\tilde{E}, \tilde{V}, \tilde{F}, \tilde{W}$  by  $\mathcal{E}, \mathcal{V}, \mathcal{F}, \mathcal{W}$  in the definition of the homotopy  $H$  used in the proof of Lemma 3.10, and to compose the resulting map from the right with a smooth homeomorphism  $\theta : [0, 4] \rightarrow [0, 4]$  with  $\theta(i) = i$ ,  $\theta^{(k)}(i) = 0$  for  $i \in \{0, 1, 2, 3, 4\}$ ,  $k \geq 1$  (to assure differentiability). Using the homotopy invariance of the virtual class, we can identify  $\{\mu\}_{GT}$  with the image of the virtual class  $[\mu_{c,\pi,\infty}]^{\text{vir}}$  in  $H_w(\mathbb{P}(\mathcal{E}), \mathbb{Z})$ . On the other hand, by the ‘‘associativity property’’ of the virtual class (see Proposition 14 (4) in [Br]) and Corollary 4.6, the latter is just the image of  $PD_{\mathbb{P}(E)}(\chi(\{\mu_{c,\pi}\}))$  via the embedding  $\mathbb{P}(E) \rightarrow \mathbb{P}(\mathcal{E})$ . But  $PD_{\mathbb{P}(E)}(\chi(\{\mu_{c,\pi}\}))$  is just a representative of  $PD_x \circ \chi_x(\{\mu\})$ . ■

## 4.2. Cohomotopy invariant jump formulae.

4.2.1. *General results.* Let

$$M \longrightarrow N \longrightarrow P$$

be a cofiber sequence of pointed  $S^1$ -spaces over a compact basis  $B$ . For every pointed  $S^1$ -space  $Y$  over  $B$  there is an associated long exact sequence of cohomotopy groups

$$\cdots \rightarrow {}_{S^1}\alpha_B^k(P, Y) \rightarrow {}_{S^1}\alpha_B^k(N, Y) \rightarrow {}_{S^1}\alpha_B^k(M, Y) \xrightarrow{\partial} {}_{S^1}\alpha_B^{k+1}(P, Y) \rightarrow \cdots \quad (21)$$

The connecting morphism

$$\partial : {}_{S^1}\alpha_B^k(M, Y) = {}_{S^1}\alpha^{k+1}(M \wedge_B S^1, Y) \longrightarrow {}_{S^1}\alpha^{k+1}(P, Y)$$

is given by composition with the contraction map  $\mathfrak{c} : P \rightarrow M \wedge_B \underline{S}^1$  induced by the fixed homotopy equivalence between  $P$  and the mapping cone of the map  $M \rightarrow N$ . For the cofiber sequence

$$S(\xi)_{+B} \longrightarrow D(\xi)_{+B} \longrightarrow \xi_B^+$$

associated with a vector bundle  $\xi$  over a compact basis  $B$ , the morphism  $\partial$  can be described in the following way. The obvious isomorphisms

$$S(\xi)_{+B} \wedge_B \underline{S}^1 \cong S(\xi) \times [0, 1] / S(\xi) \times \{0, 1\}, \quad \xi_B^+ \cong S(\xi) \times [0, 1] / \sim$$

(where  $\sim$  is the equivalence relation generated by  $(v, 0) \sim (v', 0)$ ,  $(v, 1) \sim (v', 1)$ ) allow us to use  $S(\xi) \times [0, 1] / S(\xi) \times \{0, 1\}$ ,  $S(\xi) \times [0, 1] / \sim$  as models for  $S(\xi)_{+B} \wedge_B \underline{S}^1$  and  $\xi_B^+$ . Using these models, the morphism  $\partial$  is given by composition with the contraction map

$$\mathfrak{c}_\xi : S(\xi) \times [0, 1] / \sim \longrightarrow S(\xi) \times [0, 1] / S(\xi) \times \{0, 1\} \quad (22)$$

induced by the identity of  $S(\xi) \times [0, 1]$ .

Consider now an oriented  $b$ -dimensional real vector space  $H$  and the cofiber sequence over  $B$  associated with the trivial bundle  $\underline{H} = B \times H$  over  $B$ :

$$\underline{S}(H)_{+B} \longrightarrow \underline{D}(H)_{+B} \longrightarrow \underline{H}_B^+$$

Let  $E$  be a Hermitian vector bundle over  $B$ . Taking smash product with  $S(E)_{+B}$  over  $B$  yields the following cofiber sequence over  $B$

$$S(E)_{+B} \wedge_B \underline{S}(H)_{+B} \rightarrow S(E)_{+B} \rightarrow S(E)_{+B} \wedge_B \underline{H}_B^+$$

Since  $S(E)_{+B} \wedge_B \underline{S}(H)_{+B} = [S(E) \times S(H)]_{+B}$ , the associated long exact cohomotopy sequence is

$$\begin{aligned} \cdots \rightarrow {}_{S^1}\alpha_B^{-1}(S(E)_{+B} \wedge_B \underline{H}_B^+, [F \oplus \underline{H}]_B^+) &\rightarrow {}_{S^1}\alpha_B^{-1}(S(E)_{+B}, [F \oplus \underline{H}]_B^+) \rightarrow \\ &\rightarrow {}_{S^1}\alpha_B^{-1}([S(E) \times S(H)]_{+B}, [F \oplus \underline{H}]_B^+) \xrightarrow{\partial} \\ &\rightarrow {}_{S^1}\alpha_B^0(S(E)_{+B}, [F]_B^+) \rightarrow {}_{S^1}\alpha_B^0(S(E)_{+B}, [F \oplus \underline{H}]_B^+) \rightarrow \dots \end{aligned} \quad (23)$$

Note that one has canonical base change isomorphisms

$${}_{S^1}\alpha_B^k([S(E) \times S(H)]_{+B}, [F \oplus \underline{H}]_B^+) \simeq {}_{S^1}\alpha_B^k(S(\tilde{E})_{+\tilde{B}}, [\tilde{F} \oplus \underline{H}]_B^+) . \quad (24)$$

associated with the projection

$$p : \tilde{B} = B \times S(H) \rightarrow B$$

(see [CJ] Proposition 5.37, Proposition 12.40 for the non-equivariant case).

A map  $\kappa : B \rightarrow S(H)$  defines a section  $j_\kappa^E : S(E)_{+B} \rightarrow [S(E) \times S(H)]_{+B}$  over  $B$  of the projection  $[S(E) \times S(H)]_{+B} \rightarrow S(E)_{+B}$ , so it defines a splitting of the exact sequence (23).

**Lemma 4.9.** *Let  $m \in {}_{S^1}\alpha_B^{-1}([S(E) \times S(H)]_{+B}, [F \oplus \underline{H}]_B^+)$ , and let  $\kappa_0, \kappa_1 : B \rightarrow S(H)$  be two maps. One has the identity*

$$(j_{\kappa_1}^E)^*(m) - (j_{\kappa_0}^E)^*(m) = d(\kappa_0, \kappa_1) \cdot \partial(m) ,$$

where  $d(\kappa_0, \kappa_1) \in {}_{S^1}\alpha_B^{-1}(B_{+B}, \underline{H}_B^+) = {}_{S^1}\alpha_B^{b-1}(B_{+B}, B_{+B})$  is the difference class of the maps  $\kappa_0, \kappa_1$  regarded as sections in the sphere bundle  $S(\underline{H})$ .

**Proof:** The difference class  $d(\kappa_0, \kappa_1)$  is defined by the map

$$\Delta : B_{+B} \wedge_B \underline{S}^1 = B \times [0, 1] /_{B} B \times \{0, 1\} \longrightarrow D_\epsilon(\underline{H}) /_{S_\epsilon} S_\epsilon(\underline{H}) = \underline{H}_B^+$$

induced by

$$(b, t) \mapsto \begin{cases} [(1-2t)\kappa_0(b)] & \text{for } 0 \leq t \leq \frac{1}{2} \\ [(2t-1)\kappa_1(e)] & \text{for } \frac{1}{2} \leq t \leq 1 . \end{cases} .$$

The connecting morphism  $\partial_H$  in the long exact sequence

$${}_{S^1}\alpha_B^{-1}(B_{+B}, \underline{H}_B^+) \xrightarrow{\partial_H} {}_{S^1}\alpha_B^0(B_{+B}, S(\underline{H})_{+B}) \rightarrow {}_{S^1}\alpha_B^0(B_{+B}, B_{+B}) \rightarrow {}_{S^1}\alpha_B^0(B_{+B}, \underline{H}_B^+)$$

is defined via the identifications

$${}_{S^1}\alpha_B^{-1}(B_{+B}, \underline{H}_B^+) = {}_{S^1}\alpha_B^0(B_{+B} \wedge_B \underline{S}^1, \underline{H}_B^+)$$

$${}_{S^1}\alpha_B^0(B_{+B}, S(\underline{H})_{+B}) = {}_{S^1}\alpha_B^0(B_{+B} \wedge_B \underline{S}^1, S(\underline{H})_{+B} \wedge_B \underline{S}^1) ,$$

by left composition with the contraction  $\mathfrak{c}_H : \underline{H}_B^+ \rightarrow S(\underline{H})_{+B} \wedge_B \underline{S}^1$ . The image of  $d(\kappa_0, \kappa_1)$  under  $\partial_H$  is just the difference  $\{\kappa_1\} - \{\kappa_0\} \in {}_{S^1}\alpha_B^0(B_{+B}, S(\underline{H})_{+B})$ .

One has obviously

$$(j_{\kappa_1}^E)^*(m) - (j_{\kappa_0}^E)^*(m) = m \circ (\{\kappa_1\} - \{\kappa_0\}) = m \circ \partial_H(d(\kappa_0, \kappa_1)) .$$

We know that  $\partial_H(d(\kappa_0, \kappa_1))$  is represented by  $\mathfrak{c}_H \circ \Delta$  and the connecting operator in the exact sequence (23) acts by right composition with the same contraction  $\mathfrak{c}_H$ . Therefore

$$\begin{aligned} (j_{\kappa_1}^E)^*(m) - (j_{\kappa_0}^E)^*(m) &= \partial(m) \circ d(\kappa_0, \kappa_1) = \partial(m) \circ (d(\kappa_0, \kappa_1) \cdot \{\text{id}_{B+B}\}) \\ &= (d(\kappa_0, \kappa_1) \cdot \partial(m)) \circ \{\text{id}_{B+B}\} = d(\kappa_0, \kappa_1) \cdot \partial(m) . \end{aligned}$$

Here we have used the fact that the composition multiplication  $\circ$  is  $S^1\alpha^*(B)$ -bilinear.  $\blacksquare$

This lemma has an important analogue for the groups  $\alpha^*(x)$  associated with a K-theory element  $x$ . For a compact space  $P$  we put

$$\alpha^*(P; x) = \varinjlim_{(E, F) \in x} S^1\alpha_B^*(S(E)_{+B} \wedge_B \underline{P}_{+B}, F_B^+) .$$

where the inductive limit is taken with respect to the category  $\mathcal{T}(x)$ . Using the methods used in section 2.3 for the definition of the groups  $\alpha^*(x)$ , and the results in section 5.1, we see that this inductive limit exists; it can be constructed by taking first the limit of  $S^1\alpha_B^*(S(E \oplus \mathbb{C}^n)_{+B} \wedge_B \underline{P}_{+B}, [F \oplus \mathbb{C}^n]_B^+)$  over  $n$ , and factorizing the result by the action of  $\tilde{J}(I[K^{-1}(B)]) \subset S^1\alpha^0(B)$ . The graded group  $\alpha^*(P; x)$  comes with an obvious homomorphism  $\alpha^*(P; x) \rightarrow \alpha^*(p_B^*(x))$ , where  $p_B : B \times P \rightarrow B$  is the projection on the first summand.

Taking the inductive limit of the connection morphisms  $\partial = \partial_{E, F}$  in (23) with respect to the category  $\mathcal{T}(x)$ , one gets a morphism

$$\partial_x := \varinjlim_{(E, F) \in x} \partial_{E, F} : \alpha^{b-1}(S(H); x) \longrightarrow \alpha^0(x) . \quad (25)$$

which is intrinsically associated with  $x$ .

Let  $\kappa : B \rightarrow S(H)$  be a fixed map. The system of morphisms

$$(j_\kappa^E)^* : S^1\alpha_B^*([S(E) \times S(H)]_{+B}, F_B^+) \rightarrow S^1\alpha_B^*(S(E)_{+B}, F_B^+)$$

induces a morphism  $j_\kappa^* : \alpha^*(S(H); x) \rightarrow \alpha^*(x)$ .

**Corollary 4.10.** *Let  $m \in \alpha^{b-1}(S(H); x)$ , and let  $\kappa_0, \kappa_1 : B \rightarrow S(H)$  be two maps. One has the identity*

$$(j_{\kappa_1}^E)^*(m) - (j_{\kappa_0}^E)^*(m) = d(\kappa_0, \kappa_1) \cdot \partial_x(m) .$$

**4.2.2. The universal perturbation and invariant jump formulae.** Let  $E, F$  be Hermitian vector bundles over a compact basis  $B$ , let  $V, W$  be Euclidean vector spaces, and let  $\mu : E \times V \rightarrow [F \times W]_B^+$  be an  $S^1$ -equivariant map over  $B$  satisfying the properties **P1** and **P2** (1) with  $h = 0$ . In other words, one has

$$\mu(0_y^E, v) = l(v) , \quad \forall y \in B \quad \forall v \in V ,$$

where  $l : V \xrightarrow{\cong} W_0 \subset W$  is a linear embedding. The cylinder construction cannot be applied to such a map, because  $\mu$  has vanishing points on the ‘‘heart’’  $0^E \times D^R(V)$  of any cylinder  $D_R(E) \times D_R(V)$ . We orient the orthogonal complement  $H$  of  $W_0$  in  $W$ , and we denote by  $b$  its dimension. Let  $\epsilon > 0$ . For every map  $\kappa : B \rightarrow S_\epsilon(H)$  we define the perturbation

$$\mu_\kappa : E \times V \rightarrow [F \times W]_B^+$$

by putting  $\mu_\kappa(e, v) := T_{\kappa(y)}(\mu(e, v))$  for  $e \in E_y$ . Here we denote by  $T_{\kappa(y)}$  the automorphism of  $[F \times (H \oplus W_0)]_B^+$  which extends the translation

$$(f, w) \mapsto (f, w + \kappa(y)) .$$

**Remark 4.11.** *If  $\epsilon > 0$  is sufficiently small, the map  $\mu_\kappa$  satisfies the properties **P1**, **P2** of section 3.1, so the cylinder construction applies and yields a stable class  $\{\mu_\kappa\} \in {}_{S^1}\alpha_B^{b-1}(S(E)_{+B}, F_B^+)$ .*

**Proof:** Suppose that  $\mu$  satisfies the property **P1** with constants  $C, c$ . Choose  $\epsilon < \frac{c}{2}$ . The map  $\mu_\kappa$  satisfies **P1** with constants  $C, c' := \frac{c}{2}$ , and **P2** with constant  $\epsilon_0 = \epsilon$ .  $\blacksquare$

Another important way to construct a map satisfying properties **P1**, **P2** is to let vary  $\kappa$  in the sphere  $S_\epsilon(H)$  and consider the *universal perturbation*

$$\tilde{\mu} : \tilde{E} \times V \longrightarrow \tilde{F} \times W$$

over the basis  $\tilde{B} := B \times S_\epsilon(H)$  (where  $\tilde{E} := p_B^*(E)$ ,  $\tilde{F} := p_B^*(F)$ ) which acts as  $\mu_\kappa$  over  $B \times \{\kappa\}$ . This map also satisfies properties **P1**, **P2** with the same constants as any  $\mu_\kappa$ , so that the cylinder construction applies and yields a class  $\{\tilde{\mu}\} \in {}_{S^1}\alpha_{\tilde{B}}^{b-1}(S(\tilde{E})_{\tilde{B}}, \tilde{F}_{\tilde{B}}^+)$ . Our next goal is to understand this class  $\{\tilde{\mu}\}$ . The essential point is to identify the image of  $\{\tilde{\mu}\} \in {}_{S^1}\alpha_{\tilde{B}}^{b-1}(S(\tilde{E})_{\tilde{B}}, \tilde{F}_{\tilde{B}}^+)$  under the connecting morphism  $\partial$ .

Recall from section 2.6 that  $\{o_{(E,F)}\} \in {}_{S^1}\alpha_B^0(S(E)_{+B}, F_B^+)$  is the class of the obvious pointed map  $S(E)_{+B} \rightarrow F_B^+$  over  $B$  which maps  $+_B$  to the infinity section, and  $S(E)$  to the trivial section.

**Proposition 4.12.** *(The  $\partial$ -image of the invariant of the universal perturbation) Via the obvious identification*

$${}_{S^1}\alpha_B^0(S(E)_{+B} \wedge_B \underline{H}_B^+, [F \oplus \underline{H}]_B^+) = {}_{S^1}\alpha_B^0(S(E)_{+B}, F_B^+)$$

one has

$$\partial(\{\tilde{\mu}\}) = -\{o_{(E,F)}\}.$$

**Proof:** As in section 3.1 fix  $R > C$  and  $\epsilon < \min(\epsilon_0, c') = \min(\epsilon, \frac{c}{2})$ . Let  $\tau_0 < R$  be sufficiently small such that  $\mu(e, v)$  remains finite for every  $(e, v) \in D_{\tau_0}(E) \times D_R(V)$ .

Step 1. We replace  $\tilde{\mu}|_{D_R(\tilde{E}) \times D_R(V)}$  by a map  $\tilde{\mu}_\tau$  which represents the same class  $\{\mu\}$  and coincides with the  $\kappa$ -independent map  $\mu$  outside the smaller cylinder  $D_\tau(E) \times D_R(V)$ .

Define  $\tilde{\mu}_\tau : D_R(\tilde{E}) \times D_R(V) \longrightarrow [\tilde{F} \times W]_{\tilde{B}}^+$  by the formula

$$\tilde{\mu}_\tau(e, \kappa, v) := \begin{cases} (1 - \frac{1}{\tau}\|e\|)(\kappa + l(v)) + \frac{1}{\tau}\|e\|\mu(e, v) & \text{for } 0 \leq \|e\| \leq \tau \\ \mu(e, v) & \text{for } \|e\| \geq \tau. \end{cases}$$

The maps  $\tilde{\mu}_\tau$  and  $\tilde{\mu}$  coincide on the heart  $0^{\tilde{E}} \times D_R(V)$  of the cylinder  $D_R(\tilde{E}) \times D_R(V)$  and they differ by the translation  $T_\kappa$  outside  $D_\tau(\tilde{E}) \times D_R(V)$ . We define a homotopy between  $\tilde{\mu}_\tau$  and  $\tilde{\mu}|_{D_R(\tilde{E}) \times D_R(V)}$  by putting

$$\tilde{\mu}_\tau^t(e, \kappa, v) := \begin{cases} (1-t)\tilde{\mu}_\tau(e, \kappa, v) + t\tilde{\mu}(e, \kappa, v) & \text{for } \|e\| \leq \tau \\ T_{t\kappa} \circ \mu(e, v) & \text{for } \|e\| \geq \tau. \end{cases}$$

**Claim:** If  $\tau$  is sufficiently small, then  $\|\tilde{\mu}_\tau^t\| \geq c'$  on  $\partial [D_R(\tilde{E}) \times D_R(V)]$  for every  $t \in [0, 1]$ .

The claim is not obvious only for points  $(e, v) \in D_\tau(\tilde{E}) \times S_R(V)$ . One has the identity

$$\begin{aligned} \tilde{\mu}_\tau^t(e, \kappa, v) &= (1-t) \left\{ \left( 1 - \frac{1}{\tau} \|e\| \right) \kappa + l(v) + \frac{1}{\tau} \|e\| [\mu(e, v) - l(v)] \right\} + t\mu(e, v) + t\kappa = \\ &= l(v) + \left( 1 - \frac{1-t}{\tau} \|e\| \right) \kappa + \left[ t + \frac{(1-t)}{\tau} \|e\| \right] [\mu(e, v) - l(v)] . \end{aligned}$$

The first two terms belong to orthogonal complements, so for  $e \in D_\tau(E)$  one has

$$\|\tilde{\mu}_\tau^t(e, \kappa, v)\| \geq \|l(v)\| - \|\mu(e, v) - l(v)\| .$$

Since  $\mu(0_y^E, v) = l(v)$ , and  $\mu$  is fiberwise differentiable with globally continuous derivatives on  $E \times V$ , one has

$$\lim_{\tau \rightarrow 0} \left\{ \sup \{ \|\mu(e, v) - l(v)\| \mid 0 \leq \|e\| \leq \tau, \|v\| \leq R \} \right\} = 0 .$$

On the other hand, for  $\|v\| = R$  one has  $\|l(v)\| = \|\mu(0_y^E, v)\| > c$ . This proves the claim.

Using the Claim and  $\|\tilde{\mu}_\tau^t(e, \kappa, v)\| = \|\kappa\| = \epsilon > 0$  we see that  $(\tilde{\mu}_\tau^t)_{t \in [0,1]}$  defines a homotopy between  $\tilde{\mu}_\tau$  and  $\tilde{\mu}|_{D_R(\tilde{E}) \times D_R(V)}$  in the space of maps for which the cylinder construction applies. Therefore

$$\{\tilde{\mu}\} = \{\tilde{\mu}_\tau\} \in S^1 \alpha_B^{b-1}([S(E) \times S(H)]_{+B}, F_B^+) \text{ for all sufficiently small } \tau > 0. \quad (26)$$

Step 2. We compute the class  $-\partial(\{\tilde{\mu}_\tau\})$ .

Regard  $\{\tilde{\mu}_\tau\}$  as an element in the group

$$S^1 \alpha_B^{-1}([S(E) \times S(H)]_{+B}, [F \oplus \underline{H}]_B^+) = S^1 \alpha_B^0([S(E)_{+B} \wedge_B \underline{S}(H)_{+B} \wedge_B \underline{S}^1, [F \oplus \underline{H}]_B^+) .$$

As explained at the beginning of this section the morphism  $\partial$  is induced by composition with the contraction map

$$\mathbf{c}_H : H^+ = S_\epsilon(H) \times [0, R] / \sim_H \rightarrow S(H)_+ \wedge S^1 = S_\epsilon(H) \times [0, R] / S(H) \times \{0, R\}$$

induced by the identity of  $S_\epsilon(H) \times [0, R]$ . The morphism  $-\partial$  is defined by composition with  $\mathbf{c}'$ , where  $\mathbf{c}'$  is induced by the map  $(\kappa, \rho) \rightarrow (\kappa, R - \rho)$ .

The class  $\{\tilde{\mu}_\tau\}$  is represented by the map

$$\tilde{m}_\tau : S(E) \times S_\epsilon(H) \times [0, R] \times D_R(V) \longrightarrow [F \times W]_B^+ / [F \times W]_B^+ \setminus D_\epsilon(F \times W)$$

given by

$$\tilde{m}_\tau(e, \kappa, \rho, v) = [\tilde{\mu}_\tau(\rho e, \kappa, v)] .$$

As we have seen in section 3.1, this map induces a map

$$S(E)_{+B} \wedge_B \underline{S}(H)_{+B} \wedge_B \underline{S}^1 \wedge_B \underline{V}_B^+ \longrightarrow [F \times W]_B^+ / [F \times W]_B^+ \setminus D_\epsilon(F \times W)$$

because it has the following properties

- (1)  $\tilde{m}_\tau(e, \kappa, 0, v)$  and  $m_\tau(e, \kappa, R, v)$  belong always to the infinity section of the right hand space,
- (2)  $\tilde{m}_\tau(e, \kappa, \rho, v)$  belongs to the infinity section of the right hand space when  $\|v\| = R$ .

The class  $-\partial(\{\tilde{\mu}_\tau\})$  is defined by the map

$$\tilde{m}'_\tau : S(E) \times S_\epsilon(H) \times [0, R] \times D_R(V) \rightarrow [F \times W]_B^+ / [F \times W]_B^+ \setminus D_\epsilon(F \times W)$$

given by

$$\tilde{m}'_\tau(e, \kappa, \rho, v) = \tilde{m}_\tau(e, \kappa, R - \rho, v) .$$

This map descends to a map

$$S(E)_{+B} \wedge_B \underline{H}_B^+ \wedge_B \underline{V}_B^+ \rightarrow [F \times W]_B^+ / [F \times W]_B^+ \setminus D_\epsilon(F \times W)$$

because it has the following properties:

- (1)  $\tilde{m}'_\tau(e, \kappa, 0, v)$  and  $\tilde{m}'_\tau(e, \kappa, R, v)$  are independent of  $\kappa$ .
- (2)  $\tilde{m}'_\tau(e, \kappa, R, v)$  belongs always to the infinity section of the right hand space.
- (3)  $\tilde{m}'_\tau(e, \kappa, \rho, v)$  belongs to the infinity section of the right hand space when  $\|v\| = R$ .

These three conditions characterize the maps of pointed spaces over  $B$  defined on  $S(E) \times S_\epsilon(H) \times [0, R] \times D_R(V)$  which descend to  $S(E)_{+B} \wedge_B \underline{H}_B^+ \wedge_B \underline{V}_B^+$ .

Step 2 (a). We deform the map  $\tilde{m}'_\tau$  in the space of maps satisfying the three properties above, by composing it with a 1-parameter family of contractions in the  $\rho$ -direction.

For  $t \in [0, 1]$  define the map

$$[\tilde{m}'_\tau]^t : S(E) \times S_\epsilon(H) \times [0, R] \times D_R(V) \rightarrow [F \times W]_B^+ / [F \times W]_B^+ \setminus D_\epsilon(F \times W)$$

by

$$[\tilde{m}'_\tau]^t(e, \kappa, \rho, v) = \tilde{m}_\tau\left(e, \kappa, (1 - t + t \frac{\tau}{R})(R - \rho), v\right) .$$

The family  $([\tilde{m}'_\tau]^t)_{t \in [0, 1]}$  defines a homotopy in the space of maps satisfying properties (1), (2), (3) above. The main point in checking (1) is the fact that the map  $\tilde{m}_\tau$  is constant with respect to  $\kappa$  for  $\rho \in [\tau, R]$ . Therefore it holds

$$-\partial(\{\tilde{\mu}_\tau\}) = \{[\tilde{m}'_\tau]^0\} = \{[\tilde{m}'_\tau]^1\} .$$

Putting  $\tilde{m}''_\tau := [\tilde{m}'_\tau]^1$ , one has

$$\begin{aligned} \tilde{m}''_\tau(e, \kappa, \rho, v) &= \tilde{m}_\tau\left(e, \kappa, \frac{\tau}{R}(R - \rho), v\right) = \left[1 - \frac{R - \rho}{R}\right] (\kappa + l(v)) + \frac{R - \rho}{R} \mu\left(\tau \frac{R - \rho}{R} e, v\right) \\ &= \frac{\rho}{R} \kappa + l(v) + \frac{R - \rho}{R} \left(\mu\left(\tau \frac{R - \rho}{R} e, v\right) - l(v)\right) , \end{aligned}$$

Step 2 (b). We remark that the family of maps  $\tilde{m}''_\tau$  has a uniform limit as  $\tau \rightarrow 0$  and we compute this limit explicitly .

Using arguments as in the proof of the claim above, we see that

$$\lim_{\tau \rightarrow 0} \frac{R - \rho}{R} \left(\mu\left(\tau \frac{R - \rho}{R} e, v\right) - l(v)\right) = 0$$

uniformly. Therefore  $\tilde{m}'' := \lim_{\tau \rightarrow 0} \tilde{m}''_\tau$  operates by  $\tilde{m}''(e, \kappa, \rho, v) = \frac{\rho}{R} \kappa + l(v)$ . It is now easy to see that the map

$$S(E)_{+B} \wedge_B \underline{H}_B^+ \wedge_B \underline{V}_B^+ \rightarrow [F \times W]_B^+ / [F \times W]_B^+ \setminus D_\epsilon(F \times W) = F_B^+ \wedge_B \underline{H}_B^+ \wedge_B [\underline{W}_0]_B^+$$

induced by  $\tilde{m}''$  is homotopic to the smash product over  $B$  of the obvious map  $S(E)_{+B} \rightarrow F_B^+$  (which represents  $o(E, F)$ ) with  $l_B^+ : V_B^+ \rightarrow [W_0]_B^+$ , and  $\text{id} : \underline{H}_B^+ \rightarrow \underline{H}_B^+$ .  $\blacksquare$

For a map  $\kappa : B \rightarrow S_\epsilon(H)$  one has

$$\{\mu_\kappa\} = (j_\kappa^E)^*(\{\tilde{\mu}\}) . \quad (27)$$

This formula shows that the individual invariant  $\{\mu_\kappa\}$  associated with a map  $\kappa : B \rightarrow S_\epsilon(H)$  is determined by the invariant associated with the universal perturbation  $\tilde{\mu}$  and the homotopy class of  $\kappa$ . Using Corollary 4.10 we obtain

**Corollary 4.13.** *(Cohomotopy invariant jump formula) One has*

$$\{\mu_{\kappa_0}\} - \{\mu_{\kappa_1}\} = o_{(E,F)} \cdot d(\kappa_0, \kappa_1) ,$$

where  $d(\kappa_0, \kappa_1) \in {}_{S^1}\alpha_B^{-1}(B_{+B}, \underline{H}_B^+)$  is the difference class of the maps  $\kappa_0, \kappa_1$  regarded as sections in the sphere bundle  $S_\epsilon(\underline{H})$ .

Suppose now that  $b = 1$ . In this case  $S_\epsilon(H)$  has two elements  $\kappa_0, \kappa_1$ , and the difference class  $d(\kappa_0, \kappa_1)$  is just the unit element of  ${}_{S^1}\alpha_B^0(B_{+B}, B_{+B})$ . Therefore, in this case, our result gives

**Corollary 4.14.** *(Cohomotopy wall crossing) Suppose  $b = 1$ . Then the two classes  $\{\mu_{\kappa_0}\}, \{\mu_{\kappa_1}\}$  associated with the two perturbations  $\mu_{\kappa_0}, \mu_{\kappa_1}$  of  $\mu$  are related by the formula*

$$\{\mu_{\kappa_0}\} - \{\mu_{\kappa_1}\} = \{o(E, F)\} .$$

We can now extend our results to the infinite dimensional case. Let  $B$  be an oriented compact manifold,  $\mathcal{E}, \mathcal{F}$  complex Hilbert bundles over  $B$ ,  $\mathcal{V}, \mathcal{W}$  real Hilbert spaces and  $\mu : \mathcal{E} \times \mathcal{V} \rightarrow \mathcal{F} \times \mathcal{W}$  an  $S^1$ -equivariant, fiberwise differentiable map over  $B$  satisfying properties  $\mathcal{P}1, \mathcal{P}3$  and  $\mathcal{P}2$  (1) with  $h = 0$ . Therefore, we have an orthogonal decomposition  $\mathcal{W} = H \oplus \mathcal{W}_0$ , and  $\mu(0_y^{\mathcal{E}}, v) = l(v)$  for every  $v \in \mathcal{V}$ , where  $l : \mathcal{V} \rightarrow \mathcal{W}_0$  is a linear isometry. We fix an orientation of the finite dimensional summand  $H$ . Defining in the same way as in the finite dimensional framework the universal perturbation  $\tilde{\mu}$ , one gets a stable class

$$\{\tilde{\mu}\} \in \alpha^*(S_\epsilon(H); x) ,$$

where  $x \in K(B)$  is the index of the complex part of the fiberwise linearization of  $\mu$  at the zero section. Using the results obtained above and taking inductive limit over  $\mathcal{T}(x)$ , we obtain

**Corollary 4.15.** (1) *The image of  $\{\tilde{\mu}\}$  under the morphism  $\partial_x$  introduced in (25) is given by*

$$\partial_x(\{\tilde{\mu}\}) = \gamma(x) .$$

(2) *Let  $\kappa_0, \kappa_1 : B \rightarrow S(H)$  two maps. Then*

$$\{\mu_{\kappa_1}\} - \{\mu_{\kappa_0}\} = d(\kappa_0, \kappa_1) \cdot \gamma(x) .$$

(3) *Suppose  $b = 1$  and write  $S_\epsilon(H) = \{\kappa_0, \kappa_1\}$ . Then*

$$\{\mu_{\kappa_1}\} - \{\mu_{\kappa_0}\} = \gamma(x) .$$



**4.3. A multiplicative property.** Let  $V_i, W_i$  be Euclidean spaces,  $E_i, F_i$  Hermitian bundles over a compact base  $B$  ( $i = 1, 2$ ) and  $\mu_i : E_i \times V_i \rightarrow [F_i \times W_i]_B^+$   $S^1$ -equivariant maps over  $B$  satisfying the properties **P1**, **P2** (1) of section 3.1. Let  $W_i = H_i \oplus W_{0,i}$  be the corresponding direct sum decompositions, and  $l_i : V_i \xrightarrow{\cong} W_{0,i}$ ,  $h_i : B \rightarrow H_i$  the maps given by **P2** (1). Fix orientations on the  $H_i$ , and put

$$V := V_1 \oplus V_2, W := W_1 \oplus W_2, H := H_1 \oplus H_2, W_0 := W_{0,1} \oplus W_{0,2}, l := l_1 \oplus l_2,$$

and consider the bundles  $E := E_1 \oplus E_2, F := F_1 \oplus F_2$ . We have a product map

$$\mu : E \times V = [E_1 \times V_1] \oplus [E_2 \times V_2] \longrightarrow [F \times W]_B^+ = [F_1 \times W_1]_B^+ \wedge_B [F_2 \times W_2]_B^+$$

over  $B$ . This map will again satisfy properties **P1**, **P2** (1) with the map

$$h = (h_1, h_2) : B \rightarrow H.$$

Note that  $\mu$  will also satisfy **P2** (2) as soon as one of the two maps  $\mu_1, \mu_2$  has this property. Suppose that  $\mu_1$  satisfies property **P2** (2) and denote by

$$\{\mu_1\} \in {}_{S^1} \alpha_B^{b_1-1}(S(E_1)_{+B}, [F_1]_B^+)$$

the corresponding stable class. The map  $\mu_2$  defines a map  $[E_2 \oplus V_2]_B^+ \longrightarrow [F_2 \oplus W_2]_B^+$  hence a class  $\{\mu_2^+\} \in {}_{S^1} \alpha_B^{b_2}([E_2]_B^+, [F_2]_B^+)$ . One can then form the product

$$\{\mu_1\} \wedge_B \{\mu_2^+\} \in {}_{S^1} \alpha_B^{b_1-1}(S(E_1)_{+B} \wedge_B [E_2]_B^+, F_B^+).$$

Consider now the contraction map  ${}_1c : S(E_1 \oplus E_2)_{+B} \rightarrow S(E_1)_{+B} \wedge_B [E_2]_B^+$  introduced in section 2.3. Using the identifications

$$[E_2]_B^+ = D_R(E_2) / {}_B S_R(E_2) = E_2 / {}_B E_2 \setminus \mathring{D}_R(E_2),$$

we can use as model for the contraction  ${}_1c$  any map of the form  ${}_1c_R^{\mathfrak{R}}$  given by

$${}_1c_R^{\mathfrak{R}}(e_1, e_2) := \left[ \frac{1}{\|e_1\|} e_1, \mathfrak{R} e_2 \right], \quad (\mathfrak{R} \geq R).$$

**Proposition 4.16.** *It holds  $\{\mu\} = {}_1c^* (\{\mu_1\} \wedge_B \{\mu_2^+\})$ .*

**Proof:** The class  $\{\mu\}$  is represented by the map of pairs

$$\begin{aligned} \mu_R : (S(E) \times [0, R] \times D_R(V), S(E) \times ([0, R] \times S_R(V) \cup \{0, R\} \times D_R(V))) \longrightarrow \\ \longrightarrow ([F \times W]_B^+, [F \times W]_B^+ \setminus D_\varepsilon(F \times W)) \end{aligned}$$

which is defined by

$$\mu_R(e_1, e_2, \rho, v_1, v_2) = [\mu_1(\rho e_1, v_1), \mu_2(\rho e_2, v_2)].$$

The class on the right  ${}_1c^* (\{\mu_1\} \wedge_B \{\mu_2^+\})$  is represented by the map  $\nu_R^{\mathfrak{R}}$  between the same pairs defined by

$$\nu_R^{\mathfrak{R}}(e_1, e_2, \rho, v_1, v_2) = \left[ \mu_1\left(\rho \frac{1}{\|e_1\|} e_1, v_1\right), \mu_2(\mathfrak{R} e_2, v_2) \right].$$

Composing  $\mu_R, \nu_R^{\mathfrak{R}}$  with the projection

$$p : [F \times W]_B^+ \longrightarrow [F \times W]_B^+ / [F \times W]_B^+ \setminus D_\varepsilon(F \times W)$$

we obtain two maps

$$m_0, m_1 : S(E) \times [0, R] \times D_R(V) \longrightarrow [F \times W]_B^+ / [F \times W]_B^+ \setminus D_\varepsilon(F \times W) \simeq [F \times W]_B^+$$

which map  $S(E) \times ([0, R] \times S_R(V) \cup \{0, R\} \times D_R(V))$  onto the infinity section in the right hand bundle. The natural homotopy between these maps is the map

$$m : [0, 1] \times S(E) \times [0, R] \times D_R(V) \longrightarrow [F \times W]_B^+ / [F \times W]_B^+ \setminus D_\varepsilon(F \times W)$$

given by

$$m_t(e_1, e_2, \rho, v_1, v_2) = \left[ \mu_1 \left( \rho \left[ 1 - t + t \frac{1}{\|e_1\|} \right] e_1, v_1 \right), \mu_2 \left( [(1-t)\rho + t\mathfrak{R}]e_2, v_2 \right) \right]$$

**Claim:** For sufficiently large  $\mathfrak{R} \geq R$  it holds

- (1) the map  $m$  is well defined and continuous at the points  $(t, e_1, e_2, \rho, v_1, v_2)$  with  $e_1 = 0$ .
- (2) the map  $m_t$  maps  $[0, 1] \times S(E) \times ([0, R] \times S_R(V) \cup \{0, R\} \times D_R(V))$  on the infinity section in the right hand bundle.

In fact we show that for  $e_2 \in [E_2]_y$ , one has

$$\lim_{u \rightarrow (t, 0_{b_1}^{E_1}, e_2, \rho, v_1, v_2)} m(u) = \infty_y ,$$

so  $m$  maps the locus  $e_2 = 0$  on the infinity section. Let  $\eta_R > 0$  be sufficiently small, such that  $\|\mu_1(e_1, v_1)\| > \varepsilon_0$  for every  $(e_1, v_1) \in D_{\eta_R}(E_1) \times D_R(V_1)$ . One has

$$\lim_{e_1 \rightarrow 0} \left\| \rho \left[ 1 - t + t \frac{1}{\|e_1\|} \right] e_1 \right\| = \rho t .$$

When  $\rho t < \eta_R$ , the first component of  $m_t(e_1, e_2, \rho, v_1, v_2)$  will already have a norm larger than  $\varepsilon_0$ . When  $\rho t \geq \eta_R$ , we compute

$$\lim_{e_1 \rightarrow 0} \|[ (1-t)\rho + t\mathfrak{R} ] e_2\| = (1-t)\rho + t\mathfrak{R} \geq \eta_R \left( \frac{1}{t} - 1 \right) + t\mathfrak{R} \geq 2\sqrt{\eta_R \mathfrak{R}} - \eta_R ,$$

which will be larger than  $R$  when  $\mathfrak{R}$  is sufficiently large. The second part of the claim is obvious.

Using the claim, it follows that  $m$  descend to an homotopy between two representatives of the classes  $\{\mu\}$  and  $c^* (\{\mu_1\} \wedge_B \{\mu_2^+\})$ .  $\blacksquare$

An interesting case is the one when  $\mu_2$  also satisfies the property **P2** (2). In this case the cylinder construction applies to  $\mu_2$  and one can write

$$\{\mu_2^+\} = \partial_2(\{\mu_2\}) ,$$

where  $\{\mu_2\} \in S^1 \alpha_B^{b_2-1}(S(E_2)_{+B}, [F_2]_B^+)$  is the invariant associated with  $\mu_2$ , and  $\partial_2$  is the connecting morphism in the long exact sequence associated with the cofiber sequence

$$S(E_2)_{+B} \longrightarrow D(E)_{+B} \longrightarrow [E_2]_B^+ .$$

Let  ${}_2c : S(E_1 \oplus E_2)_{+B} \rightarrow [E_1]_B^+ \wedge_B S(E_2)_{+B}$  be the standard contraction. In this case, our multiplication formula becomes

**Corollary 4.17.** *Suppose that both maps  $\mu_1, \mu_2$  satisfy properties **P1**, **P2**. Then*

- (1)  $\{\mu\} = {}_1c^* (\{\mu_1\} \wedge_B \partial_2(\{\mu_2\})) = {}_2c^* (\partial_1(\{\mu_1\}) \wedge_B \{\mu_2\})$ .
- (2)  $\{\mu\}$  is a 2-torsion element in  $S^1 \alpha_B^{b-1}(S(E)_{+B}, F_B^+)$ .

**Proof:** The first statement follows directly from Proposition 4.16. For the second, note that the connecting morphism  $\partial$  associated with cofiber sequence

$$[S(E_1) \times S(E_2)]_{+B} \xrightarrow{\Phi} S(E_1 \oplus E_2)_{+B} \xrightarrow{\Psi} [S(E_1)_{+B} \wedge_B [E_2]_B^+] \vee_B [[E_1]_B^+ \wedge_B S(E_2)_{+B}]$$

is induced by the contraction

$$\begin{aligned} c : \{S(E_1)_{+B} \wedge_B [E_2]_B^+\} \vee_B \{[E_1]_B^+ \wedge_B S(E_2)_{+B}\} &\longrightarrow \\ &\longrightarrow S(E_1)_{+B} \wedge_B S(E_2)_{+B} \wedge_B \underline{S}^1 = [S(E_1) \times S(E_2)]_{+B} \times_B \underline{S}^1 \end{aligned}$$

which operates as  $\text{id}_{S(E_1)_{+B}} \wedge_B \mathbf{c}_{E_2}$  on the first summand of the bouquet and as  $\mathbf{c}_{E_1} \wedge_B \text{id}_{S(E_2)_{+B}}$  on the second summand (see formula (22)). The pair  $(\{\mu_1\}, \{\mu_2\})$  defines an element  $\{\mu_1\} \wedge_B \{\mu_2\} \in {}_{S^1} \alpha_B^{b-2}([S(E_1) \times S(E_2)]_{+B}, F_B^+)$ . Since the connecting morphisms  $\partial_i$  are induced by the contractions  $\mathbf{c}_{E_i}$ , we see that

$$\partial(\{\mu_1\} \wedge_B \{\mu_2\}) = (\partial_1(\{\mu_1\}) \wedge_B \{\mu_2\}, \{\mu_1\} \wedge_B \partial_2(\{\mu_2\})).$$

Using the long exact sequence associated with our cofiber sequence, one obtains  $\Psi^* \circ \partial = 0$ , hence

$$\Psi^*(\partial_1(\{\mu_1\}) \wedge_B \{\mu_2\} + \{\mu_1\} \wedge_B \partial_2(\{\mu_2\})) = 0.$$

But is easy to see that  $\Psi^*$  acts as  ${}_2c^*$  on the first summand of the bouquet and as  ${}_1c^*$  on the second summand.  $\blacksquare$

**Remark 4.18.** Proposition 4.16 and Corollary 4.17 can be extended easily to the infinite dimensional framework of bundle maps between Hilbert bundles. The statements are similar. For Proposition 2 one first has to introduce a “weaker invariant”

$$\{\mu^+\} \in \alpha^*(x^+) := \varinjlim_{(E,F) \in x} {}_{S^1} \alpha_B^*(E_B^+, F_B^+)$$

associated with a bundle map  $\mu$  which satisfies the properties  $\mathcal{P}1$ ,  $\mathcal{P}3$ , and  $\mathcal{P}2$  (1) (but not necessarily  $\mathcal{P}2$  (2)), and to note that, in general, for two elements  $x_1, x_2 \in K(B)$  there exists a well defined natural bilinear multiplication

$$\wedge_B : \alpha^k(x_1) \times \alpha^l([x_2]^+) \rightarrow \alpha^{k+l}(x_1 + x_2).$$

## 5. APPENDIX

**5.1. Inductive limits of functors.** We recall the following important definition

**Definition 5.1.** (see [AM] p. 148) A filtering category is category  $\mathcal{C}$  with the properties

- F1. For every pair  $(O, O')$  of objects, there exists an object  $O''$  and morphisms  $O \rightarrow O'', O' \rightarrow O''$ .
- F2. For every two morphisms  $u, v : O \rightarrow O'$  there exists an object  $O''$  and a morphism  $w : O' \rightarrow O''$  such that  $w \circ u = w \circ v$ .

**Proposition 5.2.** (see [AM], p. 149-150) Let  $\mathcal{A}$  be one of the categories  $\text{Sets}$ ,  $\text{Ab}$  or  $\mathcal{G}r$ , and let  $\mathcal{C}$  be a filtering small category. Then any functor  $F : \mathcal{C} \rightarrow \mathcal{A}$  has an inductive limit, which can be constructed in the classical way: one factorizes the disjoint union  $\coprod_{O \in \text{Ob}(\mathcal{C})} F(O)$  by the equivalence relation

$$(O, x) \sim (O', x') \text{ if } \exists u : O \rightarrow O', u' : O' \rightarrow O'' \text{ with } F(u)(x) = F(u')(x'). \quad (28)$$

When  $\mathcal{A} = \text{Ab}$  or  $\mathcal{G}r$ , one endows the obtained quotient with the operation induced by the group operations on the summands  $F(O)$  of the disjoint union.

We will say that  $\mathcal{C}$  is *weakly filtering* if it satisfies F1 and the following weak form of the axiom F2.

$\tilde{F}2$ . For every two morphisms  $u, v : O \rightarrow O'$  there exists an object  $O''$  and morphisms  $w, z : O' \rightarrow O''$  such that  $w \circ u = z \circ v$ .

**Remark 5.3.** *Suppose that  $\mathcal{C}$  is weakly filtering and small. Then the relation  $\sim$  defined in (28) is still an equivalence relation, and the conclusion of Proposition 5.2 holds for  $\mathcal{A} = \mathcal{S}ets$ .*

For  $\mathcal{A} = \mathcal{A}b$  or  $\mathcal{G}$  one cannot endow the quotient of the disjoint union by this equivalence relation with a coherent group structure using only the weakly filtering condition.

**Proof:** It suffices to check that  $\sim$  is transitive. Let  $x \in F(O)$ ,  $x' \in F(O')$ ,  $x'' \in F(O'')$  with  $x \sim x'$ ,  $x' \sim x''$ . Therefore there exists morphisms  $u : O \rightarrow \hat{O}$ ,  $u' : O' \rightarrow \hat{O}$ ,  $v' : O' \rightarrow \tilde{O}$ ,  $v'' : O'' \rightarrow \tilde{O}$  such that  $F(u)(x) = F(u')(x')$  and  $F(v')(x') = F(v'')(x'')$ . By F1 there exists morphisms  $\hat{w} : \hat{O} \rightarrow O_0$ ,  $\tilde{w} : \tilde{O} \rightarrow O_0$ . We apply  $\tilde{F}2$  to the morphisms  $\hat{w}u'$ ,  $\tilde{w}v'' : O' \rightarrow O_0$ . We obtain morphisms  $\hat{z} : O_0 \rightarrow O_1$  such that  $\hat{z}\hat{w}u' = \tilde{z}\tilde{w}v''$ . Therefore

$$\begin{aligned} F(\hat{z}\hat{w}u')(x) &= F(\hat{z}\hat{w})(F(u)(x)) = F(\hat{z}\hat{w})(F(u')(x')) = F(\hat{z}\hat{w}u')(x') = \\ &= F(\tilde{z}\tilde{w}v'')(x') = F(\tilde{z}\tilde{w})(F(v')(x')) = F(\tilde{z}\tilde{w})(F(v'')(x'')) = F(\tilde{z}\tilde{w}v'')(x''), \end{aligned}$$

hence  $x \sim x''$ . ■

Unfortunately, we will need inductive limits of functors defined on index categories which are not small. In this case the disjoint union considered in Remark 5.2 might not be a set. However, there exists a simple situation when the existence of an inductive limit is guaranteed:

**Remark 5.4.** *Let  $\mathcal{C}$  be a weakly filtering category,  $Q \in \mathcal{O}b(\mathcal{C})$  a fixed object and  $F : \mathcal{C} \rightarrow \mathcal{A}$  a functor such that  $F(u)$  is surjective for every morphism  $u : Q \rightarrow O$ .*

(1) *Suppose  $\mathcal{A} = \mathcal{S}ets$ .*

(a) *The relation on  $F(Q)$  defined by*

$$x \approx x' \text{ if } \exists u, v : Q \rightarrow O \text{ such that } F(u)(x) = F(v)(x') \quad (29)$$

*is an equivalence relation. Put  $L := F(Q)/\approx$ .*

(b) *For any  $O \in \mathcal{O}b(\mathcal{C})$  there exists a unique map  $f_O : F(O) \rightarrow L$  such that  $f_O(x) = [y]$  for any pair  $(x, y) \in F(O) \times F(Q)$  such that there exist morphisms  $u : O \rightarrow \hat{O}$ ,  $v : Q \rightarrow \hat{O}$  with  $F(u)(x) = F(v)(y)$ . The system  $(f_O)_{O \in \mathcal{O}b(\mathcal{C})}$  is  $F$ -compatible (i.e. it holds  $f_{O'} \circ F(w) = f_O$  for any morphism  $u : O \rightarrow O'$ ).*

(c) *The system  $(f_O)_{O \in \mathcal{O}b(\mathcal{C})}$  satisfies the universal property of the inductive limit, so the inductive limit of  $F$  exists and can be identified with  $L$ .*

(2) *Suppose  $\mathcal{A} = \mathcal{A}b$  or  $\mathcal{G}$ .*

(a) *Let  $H$  be a smallest normal subgroup of  $F(Q)$  which contains the elements  $x'x^{-1}$  with  $x \approx x'$ . Put  $L := F(Q)/H$ .*

(b) *The system of morphism  $(f_O : F(O) \rightarrow L)_{O \in \mathcal{O}b(\mathcal{C})}$  defined in a similar way is  $F$ -compatible and satisfies the universal property of the inductive limit. Therefore the inductive limit of  $F$  exists and can be identified with  $L$ .*

**Definition 5.5.** (see [AM] p. 149) Let  $\mathcal{N}, \mathcal{C}$  be categories. A functor  $\Theta : \mathcal{N} \rightarrow \mathcal{C}$  will be called

- (1) *cofinal*, if
  - C1. For any  $O \in \mathcal{Ob}(\mathcal{C})$  there exists  $n \in \mathcal{Ob}(\mathcal{N})$  and  $u : O \rightarrow \Theta(n)$ .
  - C2. For every  $n \in \mathcal{Ob}(\mathcal{N})$ ,  $O \in \mathcal{Ob}(\mathcal{C})$ , and  $u : \Theta(n) \rightarrow O$ , there exists  $m \in \mathcal{Ob}(\mathcal{N})$ ,  $\nu : n \rightarrow m$  and  $v : O \rightarrow \Theta(m)$  such that  $vu = \Theta(\nu)$ .
- (2) *cofinal in the sense of Artin-Mazur* (see [AM] p. 149), if
  - C1. holds,
  - C2. For every  $O \in \mathcal{Ob}(\mathcal{C})$ ,  $n \in \mathcal{Ob}(\mathcal{N})$  and  $u, v : O \rightarrow \Theta(n)$ , there exists a morphism  $\mu : n \rightarrow m$  in  $\mathcal{N}$  such that  $\Theta(\mu)u = \Theta(\mu)v$ .

**Remark 5.6.** (1) If  $\mathcal{N}$  is filtering and  $\Theta$  is cofinal in the sense of Artin-Mazur, then  $\Theta$  is cofinal and  $\mathcal{C}$  is filtering.  
 (2) If  $\mathcal{C}$  is filtering and  $\Theta$  is cofinal, then  $\Theta$  is cofinal in the sense of Artin-Mazur.  
 (3) If  $\Theta : \mathcal{N} \rightarrow \mathcal{C}$  is cofinal, where  $\mathcal{N}, \mathcal{C}$  are both small and filtering, and  $F : \mathcal{C} \rightarrow \mathcal{A}$  is a functor (with  $\mathcal{A} = \text{Sets}, \text{Ab}$  or  $\text{Gr}$ ) then the canonical morphism

$$\varinjlim_{n \in \mathcal{Ob}(\mathcal{N})} F(\Theta(n)) \rightarrow \varinjlim_{O \in \mathcal{Ob}(\mathcal{C})} F(O)$$

is an isomorphism.

**Proof:** 1. Let  $u : \Theta(n) \rightarrow O$  be a morphism. Using C1, we can find a morphism  $w : O \rightarrow \Theta(m)$ ; since  $\mathcal{N}$  is filtering, we can find morphisms  $\eta : n \rightarrow k$ ,  $\kappa : m \rightarrow k$ . Therefore, we get two morphisms  $\Theta(\eta), \Theta(\kappa)wu : \Theta(n) \rightarrow \Theta(k)$ . By C2, there exists  $\mu : k \rightarrow l$  such that  $\Theta(\mu)\Theta(\eta) = \Theta(\mu)\Theta(\kappa)wu$ . This shows  $[\Theta(\mu\kappa)w]u = \Theta(\mu\eta)$ , so C2 holds with  $v = \Theta(\mu\kappa)w$  and  $\nu = \mu\eta$ . The fact that  $\mathcal{C}$  is filtering is stated in [AM] p. 149.

2. Let  $u, v : O \rightarrow \Theta(n)$  be two morphisms. Since  $\mathcal{C}$  is filtering, there exists  $w : \Theta(n) \rightarrow O'$  with  $wu = vw$ . By C2, we can find  $m \in \mathcal{Ob}(\mathcal{N})$ ,  $\nu : n \rightarrow m$  and  $v' : O' \rightarrow \Theta(m)$ , such that  $v'w = \Theta(\nu)$ . We will have  $\Theta(\nu)u = v'wu = v'vw = \Theta(\nu)v$ , which proves C2.

3. See Proposition 1.8 p. 150 in [AM]. ■

**Example 1.** Let  $B$  be a compact space and let  $\mathcal{U}_B$  be the category of complex bundles over  $B$ . A morphism  $U \rightarrow U'$  is a pair  $u = (i, U_1)$  consisting of a bundle embedding  $i : U \rightarrow U'$  and a complement  $U_1$  of  $i(U)$  in  $U'$  (see section 2.3). The category  $\mathcal{U}_B$  satisfies F1 but not F2, so is *not* filtering. Let  $\mathcal{N}$  be category associated with the ordered set  $(\mathbb{N}, \leq)$ . Then the obvious functor  $\Theta : \mathcal{N} \rightarrow \mathcal{U}_B$  which associates to  $n$  the trivial bundle  $\mathbb{C}^n$  and to an inequality  $n \leq m$  the standard morphism  $\mathbb{C}^n \rightarrow \mathbb{C}^m$  is cofinal. This follows from the fact that any vector bundle on  $B$  possesses a complement  $\xi'$ . Note however that  $\Theta$  is not cofinal in the sense of Artin-Mazur.

**Example 2.** For a category  $\mathcal{C}$  and object  $Q \in \mathcal{Ob}(\mathcal{C})$  we will denote by  $\mathcal{C}_Q$  the category whose objects are morphism  $u : Q \rightarrow O$  and whose morphisms are

$$\text{Hom}(Q \xrightarrow{u} O, v : Q \xrightarrow{v} O') := \{w : O \rightarrow O' \mid w \circ u = v\}.$$

A morphism  $u : Q \rightarrow Q'$  induces in an obvious way a pull-back functor  $u^* : \mathcal{C}_{Q'} \rightarrow \mathcal{C}_Q$ . If  $\mathcal{C}$  is filtering then  $\mathcal{C}_Q$  is filtering and the target functor  $T : \mathcal{C}_Q \rightarrow \mathcal{C}$  is both cofinal and cofinal in the sense of Artin-Mazur.

**Definition 5.7.** *A stable category is a pair  $(\mathcal{U}, A)$ , where  $\mathcal{U}$  is a category and  $A : \mathcal{U} \rightarrow \mathcal{G}r$  a functor, such that*

F1. *holds in  $\mathcal{U}$ .*

S1.  *$A(O) = \text{Aut}(O)$  for every  $O \in \mathcal{O}b(\mathcal{C})$ .*

S2. *For any  $u : O \rightarrow O'$  and  $a \in \text{Aut}(O)$  one has  $A(u)(a) \circ u = u \circ a$*

S3. *For every two morphisms  $u, v : O \rightarrow O'$  in  $\mathcal{U}$  there exists an object  $O''$ , a morphism  $w : O' \rightarrow O''$  and  $a \in A(O'')$  such that  $a \circ w \circ u = w \circ v$ .*

Note that if  $(\mathcal{U}, A)$  is a stable category, then  $\mathcal{U}$  is weakly filtering (use S3).

**Example 3.** Defining the automorphism push-forward functors in the obvious way, the categories  $\mathcal{U}_B, \mathcal{C}_B, \mathcal{T}(x)$  introduced in this article can be regarded as stable categories.

Let  $(\mathcal{U}, A)$  be a stable category,  $Q \in \mathcal{O}b(\mathcal{U})$  a fixed object and  $F : \mathcal{U} \rightarrow \mathcal{A}b$  a functor such that  $F(u)$  is an isomorphism for any morphism  $u : Q \rightarrow O$ . We know by Remark 5.4 that the inductive limit of  $F$  will be a quotient of  $F(Q)$ . We need an explicit description of this quotient. For every object  $u : Q \rightarrow O$  in the category  $\mathcal{U}_Q$  we note that the group  $A(T(u))$  acts on  $F(Q)$  via the isomorphism  $F(u) : F(Q) \rightarrow F(T(u))$ . A morphism  $w : T(u) \rightarrow T(v)$  can be regarded as an element in  $\text{Hom}_{\mathcal{U}_Q}(u, v)$  and defines a group morphism  $A(w) : A(T(u)) \rightarrow A(T(v))$  which intertwines the actions of these groups on  $F(Q)$ .

**Proposition 5.8.** *Let  $(\mathcal{U}, A)$  be a stable category  $Q \in \mathcal{O}b(\mathcal{U})$  a fixed object, and  $F : \mathcal{U} \rightarrow \mathcal{A}b$  a functor such that  $F(u)$  is an isomorphism for any  $u : Q \rightarrow O$ . Let  $\mathcal{N}$  be a small filtering category and  $\Theta : \mathcal{N} \rightarrow \mathcal{U}_Q$  functor satisfying the cofinality axiom C1. Put*

$$\mathbb{A} := \varinjlim_{n \in \mathcal{O}b(\mathcal{N})} A(T(\Theta(n))) ,$$

and note that  $\mathbb{A}$  acts on  $F(Q)$  in a natural way. The inductive limit  $\varinjlim_{O \in \mathcal{O}b(\mathcal{U})} F(O)$  exists and can be identified with the quotient  $F(Q)/I[\mathbb{A}]F(Q)$ .

**Proof:** We know by Remark 5.4 that the inductive limit of  $G$  exists and can be identified with the quotient  $F(Q)/H$  where  $H$  is the group generated by the elements of the form  $x - x'$  where  $x, x' \in F(Q)$  are such that there exists  $u, u' : Q \rightarrow O$  with  $F(u)(x) = F(u')(x')$ . We claim that the set of such pairs  $(x, x')$  coincides with the set of pairs of the form  $(\mathfrak{a}x', x')$  with  $x' \in F(Q)$ ,  $\mathfrak{a} \in \mathbb{A}$ .

Indeed, if  $F(u)(x) = F(u')(x')$ , choose  $v : O \rightarrow \hat{O}$  and  $a \in A(\hat{O})$  such that  $vu' = avu$ . The morphism  $vu$  can be regarded as an object in the category  $\mathcal{U}_Q$ . Since  $\Theta$  satisfies the axiom C1, there exists  $n \in \mathcal{O}b(\mathcal{N})$  and a morphism  $vu \rightarrow \Theta(n)$  in  $\mathcal{U}_Q$ , i.e. a morphism  $w : \hat{O} \rightarrow T(\Theta(n))$  such that  $wvu = \Theta(n)$ . We obtain

$$\begin{aligned} F(\Theta(n))(x) &= F(wvu)(x) = F(wvu')(x') = F(wavu)(x') = F(A(w)(a)wvu)(x') = \\ &= A(w)(a)(F(wvu)(x')) = A(w)(a)(F(\Theta(n))(x')) , \end{aligned}$$

which shows that  $x = \mathfrak{a}x'$ , where  $\mathfrak{a}$  is the class of  $A(w)(a) \in A(T(\Theta(n)))$  in  $\mathbb{A}$ . Conversely let  $\mathfrak{a} = [a] \in \mathbb{A}$  be represented by  $a \in A(T(\Theta(n)))$  and suppose that  $x =$

$\alpha x'$ . This means  $F(\Theta(n))(x) = a(F(\Theta(n))(x'))$  so, putting  $u := \Theta(n)$ ,  $u' := a\Theta(n)$  one will have  $F(u)(x) = F(u')(x')$ .  $\blacksquare$

Let  $(\mathcal{U}, A)$  be a stable category and let  $G : \mathcal{C} \rightarrow \mathcal{A}$  be a functor, where  $\mathcal{A}$  is one of the categories  $\mathit{Sets}$ ,  $\mathit{Gr}$  or  $\mathit{Ab}$ .

**Definition 5.9.** *We will say that the stabilized automorphisms act trivially on  $G$  (or that  $G$  satisfies the trivial stable action axiom) if*

TSA. *For every  $O \in \mathcal{Ob}(\mathcal{C})$ ,  $x \in G(O)$  and  $a \in A(O)$  there exists a morphism  $u : O \rightarrow O'$  such that  $G(u)(G(a)(x)) = G(u)(x)$ .*

*In the presence of functor  $\Theta : \mathcal{N} \rightarrow \mathcal{U}$ , we say that the  $\Theta$ -stabilized automorphisms act trivially on  $G$  (or that  $G$  satisfies the trivial  $\Theta$ -stable action axiom) if*

$\Theta$ SA. *For every  $n \in \mathcal{Ob}(\mathcal{N})$ ,  $x \in G(\Theta(n))$  and  $a \in A(\Theta(n))$  there exists a morphism  $\nu : n \rightarrow m$  such that  $G(\Theta(\nu))(G(a)(x)) = G(\Theta(\nu))(x)$ .*

**Remark 5.10.** *If  $\Theta$  is cofinal and  $G$  satisfies  $\Theta$ SA, then it also satisfies TSA. If  $\mathcal{U}$  is filtering, then any functor  $G : \mathcal{C} \rightarrow \mathcal{A}$  satisfies the axiom TSA. If, moreover,  $\Theta$  is cofinal, then  $G$  also satisfies  $\Theta$ SA.*

Let  $(\mathcal{U}, A)$  be a stable category, and let  $G : \mathcal{U} \rightarrow \mathcal{A}$  be a functor. Let  $\mathcal{N}$  be a *small filtering* category and  $\Theta : \mathcal{N} \rightarrow \mathcal{U}$  a cofinal functor such that  $\Theta$ SA holds. Consider the classical inductive limit  $L_\Theta := \varinjlim_{n \in \mathcal{Ob}(\mathcal{N})} G(\Theta(n))$ . For every  $O \in \mathcal{Ob}(\mathcal{U})$

we define a morphism  $f_O : G(O) \rightarrow L_\Theta$  by  $f_O(x) := [G(v)(x)]$  where  $v : O \rightarrow \Theta(n)$  is a morphism (whose existence is guaranteed by C1).

**Proposition 5.11.** *Under the assumptions and with the notations above it holds*

- (1) *For any  $O \in \mathcal{Ob}(\mathcal{U})$  the map  $f_O$  is well defined. The system of maps  $(f_O)_{O \in \mathcal{Ob}(\mathcal{U})}$  is  $G$ -compatible i.e. for any  $u : O \rightarrow O'$  on has  $f_{O'} \circ G(u) = f_O$ . When  $\mathcal{A} = \mathit{Ab}$  or  $\mathit{Gr}$ , the map  $f_O$  is a group morphism.*
- (2) *The system  $(f_O)_{O \in \mathcal{Ob}(\mathcal{U})}$  satisfies the universal property of the inductive limit, so the functor  $G$  admits an inductive limit in  $\mathcal{A}$  which can be identified with  $L_\Theta$ .*

We agree to write  $u(x)$ ,  $v(x)$  ..., instead of  $G(u)(x)$ ,  $G(v)(x)$  ..., to save on notations.

**Proof:** 1. Let  $v : O \rightarrow \Theta(n)$ ,  $v' : O \rightarrow \Theta(n')$  two morphisms. Since  $\mathcal{N}$  is filtering, there exists morphisms  $\nu : n \rightarrow m$ ,  $\nu' : n' \rightarrow m$ . We apply the axiom S3 to the morphisms  $\Theta(\nu)v$ ,  $\Theta(\nu')v'$  and we get a morphism  $w : \Theta(m) \rightarrow \hat{O}$  and  $a \in A(\hat{O})$  such that  $w\Theta(\nu')v' = aw\Theta(\nu)v$ . We apply the axiom C2 to  $w$  and we get morphisms  $u : \hat{O} \rightarrow \Theta(k)$ ,  $\mu : m \rightarrow k$  such that  $uw = \Theta(\mu)$ . We have

$$\Theta(\mu\nu')v' = uw\Theta(\nu')v' = uaw\Theta(\nu)v = A(u)(a)uw\Theta(\nu)v = A(u)(a)\Theta(\mu\nu)v .$$

Using the axiom  $\Theta$ SA we obtain a morphism  $\eta : k \rightarrow l$  such that

$$\Theta(\eta) [A(u)(a)\Theta(\mu\nu)v(x)] = \Theta(\eta) [\Theta(\mu\nu)v(x)] .$$

Therefore  $\Theta(\eta\mu\nu')(v'(x)) = \Theta(\eta\mu\nu)(v(x))$ , which show that  $v(x)$  and  $v'(x')$  define the same element in  $L_\Theta$ . The second claim consider an element  $x \in G(O)$  write  $f_{O'}(u(x)) = v'(u(x)) = (v'u)(x)$  for morphisms  $v' : O' \rightarrow \Theta(n')$ . It suffices to notice that  $(v'u)(x)$  is a representative of  $f_O(x)$ . The third claim is obvious.

2. Let  $\Lambda \in \mathcal{Ob}(\mathcal{A})$  and  $(g_O)_{O \in \mathcal{Ob}(\mathcal{U})}$ ,  $g_O : G(O) \rightarrow \Lambda$  a system of  $G$ -compatible

morphisms. Using the system  $(g_{\Theta(n)})_{n \in \mathcal{O}b(\mathcal{N})}$  (which is  $G \circ \Theta$ -compatible) we get a unique morphism  $g : L_{\Theta} \rightarrow \Lambda$  such that  $g \circ c_n = g_{\Theta(n)}$  for every  $n \in \mathcal{O}b(\mathcal{N})$ , where  $c_n : G(\Theta(n)) \rightarrow L_{\Theta}$  is the canonical morphism. It remains to prove that  $g \circ f_O = g_O$  for every  $O \in \mathcal{O}b(\mathcal{U})$ . Let  $x \in G(O)$  and choose  $v : G(O) \rightarrow \Theta(n)$ . One has

$$g \circ f_O(x) = g(c_n(v(x))) = g_{\Theta(n)}(v(x)) = g_O(x).$$

■

**5.2. Pointed bundle maps between sphere bundles.** Let  $X$  be a CW complex and  $Y \subset X$  a subcomplex. For two sections  $s', s''$  in an oriented  $r$ -sphere bundle over a CW complex  $X$  which coincide over  $Y$ , we denote by  $o(s', s'') \in H^r(X, Y, \mathbb{Z})$  the *primary obstruction* to the existence of a homotopy between  $s'$  and  $s''$  in the space of sections which coincides with  $s'|_Y = s''|_Y$  on  $Y$  [St].

Let  $\pi_{\zeta} : \zeta \rightarrow \mathcal{B}$  be an oriented real bundle of rank  $r$  over a CW complex  $\mathcal{B}$ . Denote by  $\pi_{\zeta}^+ : \zeta_{\mathcal{B}}^+ =: \hat{\mathcal{B}} \rightarrow \mathcal{B}$  the bundle projection of the associated sphere bundle, and consider the pull-back bundle  $\hat{\zeta} := [\pi_{\zeta}^+]^*(\zeta)$  on  $\hat{\mathcal{B}}$ . The sphere bundle  $\hat{\zeta}_{\hat{\mathcal{B}}}^+ = [\pi_{\zeta}^+]^*(\zeta_{\mathcal{B}}^+)$  comes with a tautological section  $\theta_{\zeta}$  and an “infinite” section  $s_{\hat{\zeta}}^{\infty}$ . These sections coincide on the subspace  $\infty_{\zeta} \subset \hat{\mathcal{B}}$ . We endow the space  $\hat{\mathcal{B}}$  with a CW structure in the following way: First, on the subspace  $\infty_{\zeta}$  we copy the CW structure from the base  $\mathcal{B}$  via  $s_{\hat{\zeta}}^{\infty}$ . Second, for every  $k$  cell  $e \subset \mathcal{B}$  we put  $\hat{e} := \pi_{\zeta}^{-1}(e)$ . The cellular chart corresponding to  $\hat{e}$  is defined in the following way: let  $u : D^k \rightarrow \bar{e} \subset \mathcal{B}$  the cellular chart of  $e$ . The pullback bundle  $u^*(\zeta)$  is trivial, so it can be identified with  $D^k \times \mathbb{R}^r = D^k \times \mathring{D}^r$ . The induced map  $D^k \times \mathring{D}^r \rightarrow \pi_{\zeta}^{-1}(\bar{e}) \subset \zeta$  can be extended to map  $\hat{u} : D^k \times D^r \rightarrow [\pi_{\zeta}^+]^{-1}(\bar{e}) \subset \zeta^+$  in an obvious way. We claim

**Lemma 5.12.** *With respect to such a cellular structure on  $\hat{\mathcal{B}}$  one has  $o(s_{\hat{\zeta}}^{\infty}, \theta) = \mathfrak{t}_{\zeta}$  in  $H^r(\hat{\mathcal{B}}, \infty_{\zeta}, \mathbb{Z})$ .*

**Proof:** Let  $P : \mathbb{E} \rightarrow \mathbb{B} := BSO(r)$  be the universal vector bundle with structure group  $SO(r)$  and a fixed CW structure on the classifying space  $\mathbb{B}$ . Since  $H^r(\hat{\mathbb{B}}, \infty_{\mathbb{E}}, \mathbb{Z}) \simeq H^0(\mathbb{B}, \mathbb{Z}) \simeq \mathbb{Z}$ , there exists an integer  $N$  such that  $o(s_{\hat{\mathbb{E}}}^{\infty}, \theta_{\mathbb{E}}) = N\mathfrak{t}_{\mathbb{E}}$ . Let  $f : \mathcal{B} \rightarrow \mathbb{B}$  a cellular map which induces the bundle  $\zeta$ . This map is covered by a bundle map  $\hat{f} : \hat{\mathcal{B}} \rightarrow \hat{\mathbb{B}}$ , which is obviously cellular and maps the subcomplex  $\infty_{\zeta}$  of  $\hat{\mathcal{B}}$  into the subcomplex  $\infty_{\mathbb{E}}$  of  $\hat{\mathbb{B}}$ . Using the functorial properties of the relative obstruction class and of the Thom class, we obtain  $o(s_{\hat{\zeta}}^{\infty}, \theta) = N\mathfrak{t}_{\zeta}$ . The integer  $N$  can be computed using any bundle  $\zeta$ , so we will choose the bundle  $\mathbb{R}^r \rightarrow \{*\}$ . The tautological section is just the identity of  $[\mathbb{R}^r]^+$ . It's easy to see that both classes can be identified with the generator of  $H^r([\mathbb{R}^r]^+, \infty, \mathbb{Z})$ . ■

**Corollary 5.13.** *Let  $\zeta$  be an oriented  $r$ -bundle over a CW complex  $\mathcal{B}$ , and let  $s$  be a section in  $\zeta_{\mathcal{B}}^+$  which coincides with  $s_{\zeta}^{\infty}$  on a subcomplex  $\mathcal{A} \subset \mathcal{B}$ . Then  $o(s_{\zeta}^{\infty}, s) = s^*(\mathfrak{t}_{\zeta})$  in  $H^r(\mathcal{B}, \mathcal{A}, \mathbb{Z})$ .*

**Proof:** Note that, with respect to the cellular decomposition of  $\hat{\mathcal{B}}$  considered above, the section  $s : \mathcal{B} \rightarrow \hat{\mathcal{B}}$  is a cellular map and maps to subcomplex  $\mathcal{A}$  into the subcomplex  $\infty_{\zeta}$ . It suffices to apply the functorial property of the relative obstruction classes with respect to cellular maps. ■



**Corollary 5.14.** *Let  $\zeta$  be an oriented  $r$ -bundle over a finite CW complex  $\mathcal{B}$  of dimension  $n \leq r$  and let  $\mathcal{A} \subset \mathcal{B}$  be a subcomplex. The map  $o : s \mapsto s^*(t_\zeta)$  defines a bijection between the set  $\Gamma_{\mathcal{A}}(\zeta_{\mathcal{B}}^+)$  of homotopy classes of sections in  $\zeta_{\mathcal{B}}^+$  which coincide with  $s_\zeta^\infty$  on  $\mathcal{A}$ , and  $H^r(\mathcal{B}, \mathcal{A}, \mathbb{Z})$ .*

**Proof:** Injectivity: for a section  $s \in \Gamma_{\mathcal{A}}(\zeta_{\mathcal{B}}^+)$  the only obstruction to the existence of a homotopy between  $s_\zeta^\infty$  and  $s$  is the primary obstruction  $o(s_\zeta^\infty, s)$  (because  $\dim(\mathcal{B}) = r$ ). To prove surjectivity consider, for any  $r$ -cell  $e \subset \mathcal{B} \setminus \mathcal{A}$ , a section  $s_e$  which coincides with  $s_\zeta^\infty$  on  $\bar{e} \setminus e$  and has a single vanishing point, which is non-degenerate. The pull-back  $s_e^*(t_\zeta)$  is a generator of  $H^r(\mathcal{B}, \mathcal{B} \setminus e) \simeq \mathbb{Z}$ . ■

**Corollary 5.15.** *Let  $\zeta_0, \zeta_1$  be two oriented bundles of ranks  $r_0, r_1$  over a finite  $n$ -dimensional complex  $\mathcal{B}$ .*

- (1) *If  $n + r_0 < r_1$  any pointed bundle map  $f : [\zeta_0]_{\mathcal{B}}^+ \rightarrow [\zeta_1]_{\mathcal{B}}^+$  over  $\mathcal{B}$  is homotopic (in the space of pointed bundle maps over  $\mathcal{B}$ ) to the fiberwise constant map  $f^\infty$  which maps  $[\zeta_0]^+$  into  $\infty_{\zeta_1}$ .*
- (2) *If  $n + r_0 = r_1$ , then a pointed bundle map  $f : [\zeta_0]_{\mathcal{B}}^+ \rightarrow [\zeta_1]_{\mathcal{B}}^+$  over  $\mathcal{B}$  is homotopic to  $f^\infty$  if and only if the class  $h_f \in H^n(\mathcal{B}, \mathbb{Z})$  defined by the condition  $f^*(t_{\zeta_1}) = [\pi_{\zeta_0}^+]^*(h_f) \cup t_{\zeta_0}$ , vanishes. Moreover, the assignment  $f \mapsto h_f$  defines a bijection between the set of homotopy classes of pointed bundle maps  $[\zeta_0]_{\mathcal{B}}^+ \rightarrow [\zeta_1]_{\mathcal{B}}^+$  and  $H^n(\mathcal{B}, \mathbb{Z})$ .*

**Proof:** It suffices to apply Corollary 5.14 to the pull-back bundle  $\tilde{\zeta}_1 := [\pi_{\zeta_0}^+]^*(\zeta_1)$  over  $\tilde{\mathcal{B}} := [\zeta_0]_{\mathcal{B}}^+$  and to identify the space of pointed bundle maps  $[\zeta_0]_{\mathcal{B}}^+ \rightarrow [\zeta_1]_{\mathcal{B}}^+$  with the space of those sections in  $[\tilde{\zeta}_1]_{\tilde{\mathcal{B}}}^+$  which coincide with  $s_{\tilde{\zeta}_1}^\infty$  on  $\infty_{\zeta_0} \subset \tilde{\mathcal{B}}$ . Then use the Thom isomorphism  $\cdot \cup t_{\zeta_0} : H^n(\mathcal{B}, \mathbb{Z}) \rightarrow H^{r_1}(\tilde{\mathcal{B}}, \infty_{\zeta_0}, \mathbb{Z})$ . ■

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