

# HARMONIC SECTIONS IN SPHERE BUNDLES, NORMAL NEIGHBORHOODS OF REDUCTION LOCI, AND INSTANTON MODULI SPACES ON DEFINITE 4-MANIFOLDS

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ABSTRACT. In the first part of the paper we prove an existence theorem for gauge invariant  $L^2$ -normal neighborhoods of the reduction loci in the space  $\mathcal{A}_a(E)$  of oriented connections on a fixed Hermitian 2-bundle  $E$ . We use this to obtain results on the topology of the moduli space  $\mathcal{B}_a(E)$  of (non-necessarily irreducible) oriented connections, and to study the Donaldson  $\mu$ -classes globally around the reduction loci. In this part of the article we use essentially the concept of harmonic section in a sphere bundle with respect to an Euclidean connection.

Second, we concentrate on moduli spaces of instantons on definite 4-manifolds with arbitrary first Betti number. We prove strong generic regularity results which imply (for bundles with “odd” first Chern class) the existence of a *connected*, dense open set of “good” metrics for which all the reductions in the Uhlenbeck compactification of the moduli space are simultaneously regular. These results can be used to define new Donaldson type invariants for definite 4-manifolds. The idea behind this construction is to notice that, for a good metric  $g$ , the geometry of the instanton moduli spaces around the reduction loci is always the same, independently of the choice of  $g$ . The connectedness of the space of good metrics is important, in order to prove that no wall-crossing phenomena (jumps of invariants) occur. Moreover, we notice that, for low instanton numbers, the corresponding moduli spaces are a priori compact and contain no reductions at all so, in these cases, the existence of well-defined Donaldson type invariants is obvious. Note that, on the other hand, there seems to be very difficult to introduce well defined numerical Seiberg-Witten invariants for definite 4-manifolds. For instance, the construction proposed in [OT2] gives a  $\mathbb{Z}$ -valued function defined on a countable set of chambers.

The natural question is to decide whether these new Donaldson type invariants yield essentially new *differential* topological information on the base manifold, or have a purely topological nature.

## CONTENTS

0. Introduction	2
1. Harmonic sections, parallel sections	5
1.1. Harmonic sections in sphere bundles	5
1.2. A normal neighborhood of the locus of Euclidean connections with 1-dimensional kernel	6
1.3. Normal neighborhoods of the reduction loci in the space of Hermitian connections	10
1.4. Twisted reductions	12
2. The Donaldson $\mu$ -classes around the reductions and the homotopy type of $\mathcal{B}_a(E)$	13

2.1.	The universal bundle around reductions	13
2.2.	The topology of the moduli space $\mathcal{B}_a(E)$	18
3.	The instanton moduli space around the reductions	20
3.1.	An abelian gauge theoretical problem	20
3.2.	A twisted abelian gauge theoretical problem	21
3.3.	Generic regularity at the reductions	23
4.	Applications	27
4.1.	Geometric properties of instanton moduli spaces on manifolds with $b_+ = 0$	27
4.2.	New Donaldson invariants	28
5.	Appendix	31
5.1.	Metric-dependent spaces of selfdual and anti-selfdual forms	31
5.2.	Admissible metrics	32
5.3.	Analytic results	36
	References	38

## 0. INTRODUCTION

The main goal of this article is to study moduli spaces of instantons over 4-manifolds with negative definite intersection form. The vanishing of  $b_+$  has an important consequence on the geometry of the instanton moduli spaces: all line bundles admit ASD connections (with respect to any metric) hence, as soon as a rank 2-bundle  $E$  splits topologically, the corresponding instanton moduli space will always contain reductions. In other words, “one cannot get rid of reductions by perturbing the metric”. On the other hand, our main applications will concern 4-manifolds with  $b_1 > 0$ , and for such manifolds the spaces of reductions are positive dimensional. Therefore, it is very important to study carefully the *global* geometry of the moduli space of connections around the loci of reductions. This will be our first goal.

To be more precise, let  $E$  be a rank 2-Hermitian bundle on a 4-manifold  $M$ , and denote  $D := \det(E)$ ,  $d = c_1(D)$ . Consider the affine space  $\mathcal{A}_a(E)$  of connections  $A$  on  $E$  which induce a fixed connection  $a$  on  $D$ , and the moduli space  $\mathcal{B}_a(E) = \mathcal{A}_a(E)/\mathcal{G}_E$ , where  $\mathcal{G}_E$  is the gauge group  $\Gamma(SU(E))$ .

Let  $l \in H^2(X, \mathbb{Z})$  such that  $l(d - l) = c_2(E)$  and consider the set  $\lambda = \{l, d - l\}$  (which has either one or two elements). We denote by  $\mathcal{A}_a^\lambda(E)$  the subspace of connections  $A \in \mathcal{A}_a(E)$ , which are simply reducible of type  $\lambda$ , i.e. which admit only two parallel line subbundles whose Chern classes are  $l, d - l$ . Such a connection will be called  $\lambda$ -reducible.  $\mathcal{A}_a^\lambda(E)$  becomes a (locally closed) submanifold of the affine space  $\mathcal{A}_a(E)$  (after suitable Sobolev completions). Our first problem is the construction of a gauge invariant  $L^2$ -normal neighborhood of this submanifold. More precisely, we will show that, denoting by  $N^\lambda$  the  $L^2$ -normal bundle of  $\mathcal{A}_a^\lambda(E)$ , the restriction of the natural map  $\nu : N^\lambda \rightarrow \mathcal{A}_a(E)$  to a sufficiently small gauge invariant neighborhood  $\mathcal{U}^\lambda$  of the zero section is a diffeomorphism on its image. Moreover, the neighborhood  $\mathcal{U}^\lambda$  is defined by a inequality of the form  $\|\alpha\|_{L^\infty} \leq \varepsilon(A)$  (on the fiber  $N_A$ ), where the assignment  $A \mapsto \varepsilon(A)$  is gauge invariant and continuous (with respect to a sufficiently fine Sobolev topology on the space of connections).

Although this statement is very natural, the proof is not easy. The difficulty comes from the fact that infinite dimensional manifolds are not locally compact. Even the fact the  $\nu$  is injective on a neighborhood of the zero section is not trivial. The main difficulty is to characterize in a convenient way the connections which are “close” to the reducible locus  $\mathcal{A}_a^\lambda(E)$ , i.e. which are “almost”  $\lambda$ -reducible.

Our argument is based on the following idea: A reduction  $A \in \mathcal{A}_a^\lambda(E)$  admits a parallel section in the sphere bundle  $S(su(E))$ . A connection which is close to being reducible should admit an *energy minimizing harmonic section* in this sphere bundle. The precise meanings of the words “energy” and “harmonic” are the following: we associate to any connection  $A \in \mathcal{A}_a(E)$  the energy functional

$$E_A(u) := \|d_A(u)\|_{L^2}^2, \quad E_A : \Gamma(S(su(E))) \longrightarrow \mathbb{R}_{\geq 0},$$

and we agree to call the critical points of this functional  $A$ -harmonic sections. After proving these results about normal neighborhoods of reduction loci, we realized that our problem can be naturally generalized in the following way: For an Euclidean bundle  $F$  on an arbitrary compact Riemannian manifold, construct a gauge invariant  $L^2$ -normal neighborhood of the space of Euclidean connections  $d_A : A^0(F) \rightarrow A^1(F)$  with 1-dimensional kernel. Therefore, in the first section we will treat this more general problem, which is of independent interest; the results concerning reducible Hermitian connections will be easily deduced as applications, taking  $F = su(E)$ .

The second section deals with the first consequences of our existence results for gauge invariant  $L^2$ -normal neighborhoods: a global description of the universal  $SO(3)$  bundle, explicit formulae for the Donaldson  $\mu$ -classes around the reduction loci and an explicit description of the homotopy type of the whole moduli space  $\mathcal{B}_a(E)$ . This space is much more complex than its open subspace  $\mathcal{B}_a^*(E)$ , which plays a central role in classical gauge theory and whose rational homotopy type has been described in [DK]. Our description of  $\mathcal{B}_a(E)$  yields an easy method to compute the cohomology of this space using the Mayer-Vietoris exact sequence.

The third section is dedicated to the geometry of the instanton moduli space around a reduction locus. Using our existence results for normal neighborhoods, we show that *in a neighborhood of the reduction locus associated with a fixed topological splitting of the bundle, the instanton moduli problem reduces to an abelian moduli problem*, which is very much similar to the Seiberg-Witten one. These abelian equations read

$$(1) \quad \begin{cases} F_b^+ - \frac{1}{2}F_a^+ & = (\alpha \wedge \bar{\alpha})^+ \\ (d_{b^* \otimes 2 \otimes a^\vee}^*, d_{b^+ \otimes 2 \otimes a^\vee}^+) \alpha & = 0, \end{cases}$$

which are equations for a pair  $(b, \alpha)$ , where  $b$  is a connection on a fixed Hermitian line bundle  $L$  and  $\alpha \in A^1(L^{\otimes 2} \otimes D^\vee)$ . This result provides simple descriptions of the linear spaces of harmonic spaces of the deformation elliptic complexes at the reductions. It is very important to have a *global* description of these linear spaces. Similar results are stated for the loci of *twisted reductions* (i.e. instantons which are locally reducible, but globally irreducible).

Our most important technical results are obtained in the last subsection of section 3: we prove strong generic regularity theorems at the reductions. Regularity at the reductions is an old, classical problem in gauge theory (see [FU], [DK]) and one might wonder whether there are still unsolved questions on this problem. The

point is that Freed-Uhlenbeck's generic regularity result is not sufficient for our purposes. For our purposes, we need a *connected*, dense open set of good metrics (metrics for which all reductions in the Uhlenbeck compactification are regular). Our proof has two steps:

1. Define a connected, dense open set of *admissible*  $C^r$ -metrics, for which the vanishing loci of the harmonic representatives of the classes

$$2l - d, l \cdot (d - l) \leq c_2(E)$$

have good geometric properties (see section 5.2 in the Appendix). Our admissibility condition is very natural: we require that the rank of the intrinsic derivative of these harmonic representatives at any vanishing point is at least 2. The difficulty is to check that this condition defines indeed a connected, dense set of metrics.

In particular, for our admissible metrics, the vanishing loci of all these harmonic representatives have Hausdorff dimension at most 2. Note that this holds for *any*  $C^\infty$ -metric by a result of Bär [B], but this result does not appear to generalize for  $C^r$ -metrics. For a metric for which this Hausdorff dimension bound holds, the statement in Lemma 4.16 [FU] holds, making possible the second step. This stronger version of Lemma 4.16 [FU] is proved in detail in the Appendix (Corollary 5.9).

2. Regard the linear space formed by the *second* harmonic spaces at the reductions as the moduli space associated with an *abelian* moduli problem, and prove a transversality theorem for this moduli problem with respect to variations of  $g$  (in the space of admissible metrics). In other words, we will prove that the parameterized moduli space (obtained by letting the metric vary in the set of admissible metrics) is smooth away of the zero-section. Next we show that – for a bundle  $E$  with odd Chern class – the projection map from the  $\mathbb{C}^*$ -quotient of this parameterized moduli space (minus the zero-section) on the space of admissible metrics is Fredholm of *negative* index  $\leq -2$ , hence its image has connected complement.

Section 4 deals with applications of our results. First we prove a simple geometric property of a particular instanton moduli space on a 4-manifold which has the homology type of a class VII surface with  $b_2 = 2$ : the two circles of reductions belong always (for any metric!) to the same connected component. We continue with the construction of a new class of Donaldson invariants; the new invariants are defined in two situations:

First we show that the topological data can be chosen such that the rank 2-bundles  $E'$  with  $c_2(E') \leq c_2(E)$  and  $c_1(E') = c_1(E)$  are all topologically non-decomposable; in this case the Uhlenbeck compactification of  $\mathcal{M}_a^{\text{ASD}}(E)$  contains no reduction, so Donaldson invariants can be defined in the usual way. In particular, for  $4c_2(E) - c_1(E)^2 = 0$ , one obtains a moduli space of  $SO(3)$ -representations of  $\pi_1(M)$ , modulo  $SU(2)$ -conjugation.

Second, one chooses the topological data such that the bundle  $E$  splits topologically,  $4c_2(E) - c_1(E)^2 \leq 3$ , and  $c_1(E) \notin 2H^2(M, \mathbb{Z}) + \text{Tors}$ . In this case  $\mathcal{M}_a^{\text{ASD}}(E)$  is compact for any metric, and our results concerning generic regularity at the reductions apply. Therefore, for a generic metric,  $\mathcal{M}_a^{\text{ASD}}(E)^*$  is a smooth manifold with a finite number of cylindrical ends towards the reduction loci. Every end is a cylinder on a projective bundle over a  $b_1(M)$ -dimensional torus.

## 1. HARMONIC SECTIONS, PARALLEL SECTIONS

**1.1. Harmonic sections in sphere bundles.** Let  $(M, g)$  be a compact oriented Riemannian  $n$ -manifold and let  $F$  be a real rank  $r$  vector bundle on  $M$  endowed with an inner product, and denote by  $S(F)$  the unit sphere bundle of  $F$ . Let  $A$  be an Euclidean connection on  $F$ . The energy functional on the space of sections  $\Gamma(S(F))$  is defined by

$$E_A(u) = \|d_A(u)\|_{L^2}^2 = \langle d_A u, d_A u \rangle .$$

The critical points of this functional will be called  $A$ -harmonic sections.

**Proposition 1.1.** *A section  $u \in \Gamma(S(F))$  is harmonic if and only if it satisfies the Euler-Lagrange equation*

$$d_A^* d_A u - |d_A u|^2 u = 0 .$$

**Proof:** The section  $u \in \Gamma(S(F))$  is a critical point of  $E_A$  if and only if

$$\langle d_A^* d_A u, v \rangle_{L^2} = 0$$

for every  $v \in T_u(\Gamma(S(F)))$ . This happens if and only if there exists a real function  $\varphi$  such that  $d_A^* d_A u = \varphi u$ . On the other hand, the well-known identity

$$\frac{1}{2} \Delta |u|^2 = (d_A^* d_A u, u) - |d_A u|^2$$

shows that one must have  $\varphi = |d_A u|^2$ . ■

For a fixed connection  $A$  the theory of  $A$ -harmonic sections is very much similar to the theory of sphere valued harmonic maps. In particular, one has a parabolic evolution equation given by the gradient flow of the functional  $E_A$ , and using this equation and its convergence properties, one can study the existence of an  $A$ -harmonic representative in a given homotopy class of sections.

Two vectors  $a, b$  of an Euclidean vector space  $V$  define an endomorphism  $a \wedge_V b \in so(V)$  given by  $a \wedge_V b(h) = (a, h)b - (b, h)a$ . The subspace

$$\{a \wedge_V b \mid b \in V\} \subset so(V)$$

coincides with the orthogonal complement  $so(a^\perp)^\perp$  of  $so(a^\perp)$  in  $so(V)$ , and the map  $a^\perp \ni b \mapsto a \wedge_V b \in so(a^\perp)^\perp$  is an isomorphism. Similarly, for a section  $u \in A^0(F)$  and a form  $v \in A^1(F)$  we obtain a form  $u \wedge_F v \in A^1(so(F))$ .

**Lemma 1.2.** *Let  $A \in \mathcal{A}(F)$ ,  $u \in \Gamma(S(F))$  and  $A_0 := A - u \wedge_F d_A u$ . Then*

- (1) *The section  $u$  is  $A_0$ -parallel.*
- (2)  *$A_0$  is the unique connection making  $u$  parallel with  $A - A_0 \in A^1(so(u^\perp)^\perp)$ .*

**Proof:** 1. One has

$$d_{A_0} u = d_A u - (u \wedge_F d_A u)(u) = d_A u - (u, u)d_A u + (u, d_A u)u = 0 ,$$

because, since  $(u, u) \equiv 1$ , it holds  $(u, d_A u) = 0$ .

2. If  $A - A_0 \in A^1(so(u^\perp)^\perp)$ , there exists a unique form  $v \in A^1(u^\perp)$  such that  $A - A_0 = u \wedge_F v$ . The relation  $d_{A_0} u = 0$  is equivalent to  $d_A u = v$ . ■

**Proposition 1.3.** *Let  $u \in \Gamma(S(F))$  and  $A \in \mathcal{A}(F)$ . The following conditions are equivalent*

- (1) *The section  $u$  is  $A$ -harmonic.*  
(2) *Putting  $A_0 := A - u \wedge_F d_A u$ , one has  $d_{A_0}^*(A - A_0) = 0$ .*

**Proof:** We write locally  $d_A u = \nabla_{e_i}^A u e^i$  with respect to a local orthonormal frame  $(e_i)$ . Therefore, taking into account that  $u$  is  $A_0$  parallel (by Lemma 1.2) and  $(u, \nabla_{e_i}^A u) = 0$ , one has

$$\begin{aligned} d_{A_0}^*(A - A_0) &= (d_{A_0})^*(u \wedge_F d_A u) = u \wedge_F (d_{A_0})^* d_A u = u \wedge_F (d_A - u \wedge_F d_A u)^* d_A u = \\ &= u \wedge_F [d_A^* d_A u + *(u \wedge_F d_A u) * d_A u] = u \wedge_F [d_A^* d_A u + *(u \wedge_F \nabla_i^A u e^i) * \nabla_j^A u e^j] \\ &= u \wedge_F [d_A^* d_A u - (\nabla_i^A u, \nabla_j^A u) u * (e^i \wedge *e^j)] = u \wedge_F [d_A^* d_A u - |d_A u|^2 u]. \end{aligned}$$

Since  $d_A^* d_A u - |d_A u|^2 u \in A^0(u^\perp)$ , the vanishing of  $u \wedge_F [d_A^* d_A u - |d_A u|^2 u]$  is equivalent to the vanishing of  $d_A^* d_A u - |d_A u|^2 u$ .  $\blacksquare$

Using Lemma 1.2 and Proposition 1.3 we obtain:

**Corollary 1.4.** *Let  $u \in \Gamma(S(F))$  be a  $A$ -harmonic section. Then there exists a unique connection  $A_0 \in \mathcal{A}(F)$  with the properties:*

- (1)  $A - A_0 \in A^1(\text{so}(u^\perp)^\perp)$ .  
(2)  $A$  is in Coulomb gauge with respect to  $A_0$ .  
(3)  $u$  is a  $A_0$ -parallel.

**1.2. A normal neighborhood of the locus of Euclidean connections with 1-dimensional kernel.** We define the locally closed subspace

$$\mathcal{A}'(F) := \{A \in \mathcal{A}(F) \mid \dim(\ker(d_A)) = 1\} \subset \mathcal{A}(F).$$

Let  $\mathcal{S}(F)$  be the space of trivial rank 1-subbundles of  $F$ . This space can be naturally identified with  $\Gamma(S(F))/\{\pm 1\}$ , and becomes a smooth manifold after suitable Sobolev completions. Two trivial rank 1-subbundles which are sufficiently  $C^0$ -close have isomorphic complements, so they are conjugate modulo the action of the gauge group  $\text{Aut}(F) = \Gamma(SO(F))$ . In other words, the action of  $\text{Aut}(F)$  on  $\mathcal{S}(F)$  is locally transitive. One has an obvious surjective map

$$w : \mathcal{A}'(F) \rightarrow \mathcal{S}(F), \quad A \mapsto \langle \ker(d_A) \rangle,$$

where  $\langle \ker(d_A) \rangle = \text{im}(\text{ev} : \ker(d_A) \times M \rightarrow F)$  stands for the subbundle generated by the line  $\ker(d_A)$ . We will use the subscript  $(\cdot)_k$  (for  $k \in \mathbb{N}$  sufficiently large) to denote Sobolev completion with respect to the  $L_k^2$ -norm.

**Proposition 1.5.** *The subset  $\mathcal{A}'(F)_k \subset \mathcal{A}(F)_k$  is a submanifold, and the surjection  $w : \mathcal{A}'(F)_k \rightarrow \mathcal{S}(F)_{k+1}$  is a submersion.*

**Proof:** We omit Sobolev indices to save on notations. Let  $A \in \mathcal{A}'(F)$  and  $\lambda = \langle \ker(d_A) \rangle \in \mathcal{S}(F)$ . The fiber  $w^{-1}(\lambda)$  is obviously an affine subspace of  $\mathcal{A}(F)$  which can be identified with  $\mathcal{A}(u^\perp)$ . The stabilizer  $H$  of  $\lambda$  is a closed Lie subgroup  $H$  of the gauge group  $\text{Aut}(F)$  whose Lie algebra  $\mathfrak{h}$  can be identified with  $A^0(\text{so}(u^\perp))$ , so it has a topological complement  $\mathfrak{h}^\perp = A^0(\text{so}(u^\perp)^\perp)$ . The restriction of the map  $\mathfrak{h}^\perp \times w^{-1}(\lambda) \rightarrow \mathcal{A}(F)$  given by  $(h, B) \mapsto \exp(h)(B)$  to a sufficiently small open neighborhood of  $(0, A)$  is an embedding which parameterizes a neighborhood of  $A$  in  $\mathcal{A}'(F)$ . This gives the submanifold structure of  $\mathcal{A}'(F)$ . Using the local transitivity of the gauge action on  $\mathcal{S}(F)$ , it follows that  $h \mapsto \exp(h)(A)$  defines a local slice of  $w$  at  $A$ , proving that  $w$  is a submersion at  $A$ .  $\blacksquare$

The main goal of this section is to prove that

**Theorem 1.6.** *The submanifold  $\mathcal{A}'(F) \subset \mathcal{A}(F)$  has a tubular, gauge invariant,  $L^2$ -normal neighborhood.*

The idea is very simple and natural: construct a neighborhood of  $\mathcal{A}'(F)$  consisting of connections which admit an (up to sign) unique harmonic, energy minimizing section.

Let  $A_0 \in \mathcal{A}'(F)$ , and let  $u$  one of the two generators in  $\Gamma(S(F))$  of the line  $\ker(d_{A_0})$ . The splitting  $F = \langle u \rangle \oplus u^\perp$  defines an  $A_0$ -parallel splitting

$$so(F) = so(u^\perp) \oplus so(u^\perp)^\perp = so(u^\perp) \oplus u \wedge_F u^\perp ,$$

which gives an  $L^2$ -orthogonal decomposition

$$A^1(so(F)) = A^1(so(u^\perp)) \oplus d_{A_0}[A^0(so(u^\perp)^\perp)] \oplus \ker[A^1((so(u^\perp)^\perp) \xrightarrow{d_{A_0}^*} A^0(so(u^\perp)^\perp))].$$

The geometric interpretations of the three factors in the decomposition above are the following: the space  $A^1(so(u^\perp))$  is the tangent space of the fiber  $w^{-1}(\langle u \rangle)$  at  $A$ ,  $n_A := (d_{A_0})[A^0(so(u^\perp)^\perp)]$  is the normal space at  $A_0$  of this fiber in the submanifold  $\mathcal{A}'(F)$ , whereas the space

$$N_{A_0} := \ker [(d_{A_0})^* : A^1((so(u^\perp)^\perp) \rightarrow A^0(so(u^\perp)^\perp))] ]$$

is the normal space at  $A_0$  of the submanifold  $\mathcal{A}'(F)$  in the space of connections  $\mathcal{A}(F)$ .

Let  $N \rightarrow \mathcal{A}'(F)$  be the normal vector bundle of the submanifold  $\mathcal{A}'(F)$ , whose fiber over  $A_0 \in \mathcal{A}'(F)$  is just the space  $N_{A_0}$  defined above. One has a natural map  $\nu : N \rightarrow \mathcal{A}(F)$  given by

$$\nu(A_0, \alpha) = A_0 + \alpha , \quad \forall A_0 \in \mathcal{A}'(F) , \quad \alpha \in N_{A_0} ,$$

which is obviously a local isomorphism at every point of the form  $(A_0, 0)$ . The map  $\nu$  is equivariant with respect to the natural gauge actions on  $N$ ,  $\mathcal{A}'(F)$  and  $\mathcal{A}(F)$ . For  $\zeta \in A^0(so(F)) = \text{Lie}(\text{Aut}(F))$  we denote by  $\zeta^\#$  the tangent field (the infinitesimal transformation) of  $N$  associated with  $\zeta$  and by  $\zeta_\#$  the corresponding tangent field of  $\mathcal{A}(F)$ . The equivariance property of  $\nu$  implies

$$(2) \quad d(\nu)(\zeta_{(A_0, \alpha_0)}^\#) = [\zeta_\#]_{A_0 + \alpha_0} = d_{A_0 + \alpha_0}(\zeta) = d_{A_0}(\zeta) + [\alpha_0, \zeta] .$$

**Lemma 1.7.** *Let  $A_0 \in \mathcal{A}'(F)$ . There exists  $\varepsilon(A_0) > 0$  such that the differential  $d_{(A_0, \alpha_0)}\nu$  at  $(A_0, \alpha_0)$  is an isomorphism for every  $\alpha_0 \in N_{A_0}$  with  $\|\alpha_0\|_{L^\infty} \leq \varepsilon(A_0)$ . The assignment  $A_0 \mapsto \varepsilon(A_0)$  can be chosen to be continuous and gauge invariant.*

**Proof:** The submersion  $w : \mathcal{A}'(F) \rightarrow \mathcal{S}(F)$  induces a submersion  $\tilde{w} : N \rightarrow \mathcal{S}(F)$ . Fix  $(A_0, \alpha_0) \in N$  and choose  $u_0 \in \ker(d_{A_0}) \cap \Gamma(S(F))$ . The subspace

$$\{\zeta_{(A_0, \alpha_0)}^\# \mid \zeta \in A^0(so(u_0^\perp)^\perp)\}$$

is a  $\tilde{w}$ -horizontal space at  $(A_0, \alpha_0)$ , i.e. a topological complement of the vertical tangent space  $T_{(A_0, \alpha_0)}(\tilde{w}^{-1}(\langle u_0 \rangle))$ . This complement is isomorphic with  $A^0(so(u_0^\perp)^\perp)$ . A vertical tangent vector

$$v' \in T_{(A_0, \alpha_0)}(\tilde{w}^{-1}(\langle u_0 \rangle))$$

can be written as  $v' = (b, \beta)$ , where  $b \in A^1(so(u_0^\perp))$  and  $\beta \in A^1(so(u_0^\perp)^\perp)$  satisfy the equation

$$d_{A_0}^* \beta - *[b \wedge * \alpha_0] = 0 .$$

(obtained by differentiating the relation  $d_{A_0}^* \alpha_0 = 0$  in the direction  $(\dot{A}_0, \dot{\alpha}_0) = (b, \beta)$ ). Therefore, one has an isomorphism

$$T_{(A_0, \alpha_0)}(N) = \left\{ (b, \beta) \in A^1(\mathfrak{so}(u_0^\perp)) \oplus A^1(\mathfrak{so}(u_0^\perp)^\perp) \mid d_{A_0}^* \beta - *[b \wedge * \alpha_0] = 0 \right\} \oplus \oplus A^0(\mathfrak{so}(u_0^\perp)^\perp) .$$

Using (2) one obtains

$$d\nu(b, \beta, \zeta) = b + \beta + d_{A_0}(\zeta) + [\alpha_0, \zeta] .$$

The statement follows now directly from Lemma 1.8 below.  $\blacksquare$

**Lemma 1.8.** *If  $\|\alpha_0\|_{L^\infty}$  is sufficiently small, then the operator*

$$A^1(\mathfrak{so}(u_0^\perp))_k \oplus A^1(\mathfrak{so}(u_0^\perp)^\perp)_k \oplus A^0(\mathfrak{so}(u_0^\perp)^\perp)_{k+1} \xrightarrow{P} A^1(\mathfrak{so}(F))_k \oplus A^0(\mathfrak{so}(u_0^\perp)^\perp)_{k-1}$$

given by

$$(b, \beta, \zeta) \mapsto (b + \beta + d_{A_0}(\zeta) + [\alpha_0, \zeta], d_{A_0}^* \beta - *[b \wedge * \alpha_0])$$

is an isomorphism.

**Proof:** We omit as usually the Sobolev indexes. Using the decomposition

$$A^1(\mathfrak{so}(F)) = A^1(\mathfrak{so}(u_0^\perp)) \oplus A^1(\mathfrak{so}(u_0^\perp)^\perp) ,$$

the operator  $P$  can be written as

$$P = \begin{bmatrix} \text{id} & 0 & [\alpha_0, \cdot] \\ 0 & \text{id} & d_{A_0} \\ -*[\cdot \wedge * \alpha_0] & d_{A_0}^* & 0 \end{bmatrix}$$

If  $(b, \beta, \zeta) \in \ker(P)$ , one obtains easily

$$b = -[\alpha_0, \zeta] ,$$

$$(3) \quad d_{A_0}^* \beta = -d_{A_0}^* d_{A_0} \zeta = *(b \wedge * \alpha_0) = *(-[\alpha_0, \zeta] \wedge * \alpha_0) ,$$

$$\Delta_{A_0} \zeta = *[[\alpha_0, \zeta] \wedge * \alpha_0] .$$

On the other hand, since  $u_0$  is parallel and  $\dim(\ker(d_{A_0})) = 1$ , one has

$$\begin{aligned} \ker(d_{A_0} : A^0(\mathfrak{su}(u_0^\perp)^\perp) \rightarrow A^1(\mathfrak{su}(u_0^\perp)^\perp)) &= \\ = u_0 \wedge_F \ker(d_{A_0} : A^0(u_0^\perp) \rightarrow A^1(u_0^\perp)) &= \{0\} \end{aligned}$$

so the minimal eigenvalue  $\eta(A_0)$  of  $\Delta_{A_0} : A^0(\mathfrak{su}(u_0^\perp)^\perp) \rightarrow A^0(\mathfrak{su}(u_0^\perp)^\perp)$  is positive. Therefore (3) implies  $\zeta = 0$  (hence  $\ker(P) = 0$ ) as soon as  $\|\alpha_0\|_{L^\infty}^2$  is sufficiently small with respect to  $\eta(A_0)$ . For surjectivity, note that the equation

$$(4) \quad P(b, \beta, \zeta) = (c, \gamma, \eta)$$

becomes

$$b + [\alpha_0, \zeta] = c , \quad \beta + d_{A_0} \zeta = \gamma , \quad d_{A_0}^* \beta - *[b \wedge * \alpha_0] = \eta .$$

Consider first the weaker equation (for the single unknown  $\zeta$ )

$$\Delta_{A_0} \zeta - *[[\alpha_0, \zeta] \wedge * \alpha_0] = -*[c \wedge * \alpha_0] - \eta + d_{A_0}^* \gamma ,$$

which is uniquely solvable, as soon as  $\|\alpha_0\|_{L^\infty}^2$  is sufficiently small, so one gets a unique solution  $\zeta \in A^0(\mathfrak{so}(u_0^\perp)^\perp)_{k+1}$  for any given triple  $(c, \gamma, \eta)$  of Sobolev type  $(L_k^2, L_k^2, L_{k-1}^2)$ . Then set  $\beta := \gamma - d_{A_0} \zeta$ ,  $b := c - [\alpha_0, \zeta]$ , and we get a solution of the equation (4) of Sobolev type  $(L_k^2, L_k^2, L_{k+1}^2)$ .  $\blacksquare$



**Lemma 1.9.** *Let  $A_0 \in \mathcal{A}'(F)$  and  $u_0 \in \ker(d_{A_0}) \cap \Gamma(S(F))$ . There exists  $\varepsilon(A_0) > 0$  such that for every  $\alpha \in A^1(\text{so}(u_0^\perp)^\perp)$  with  $d_{A_0}^*(\alpha) = 0$  and  $\|\alpha\|_{L^\infty} \leq \varepsilon(A_0)$  the energy functional  $E_{A_0+\alpha}$  on the space  $\Gamma(S(F))$  obtains its absolute minimum at  $\pm u_0$  and only at these sections. Moreover, the assignment  $\mathcal{A}'(F) \ni A_0 \mapsto \varepsilon(A_0)$  can be chosen to be continuous and gauge invariant.*

**Proof:** Consider a section  $u \in \Gamma(S(F))$ , put  $v := u - u_0$ ,  $v' := u + u_0$ , and set  $\alpha := u_0 \wedge_F a$ , where  $a \in A^1(u_0^\perp)$ . This implies  $|\alpha|^2 = 2|a|^2$ . One has

$$\begin{aligned} \|d_{A_0+\alpha}u\|^2 - \|d_{A_0+\alpha}u_0\|^2 &= \|d_{A_0+\alpha}v\|^2 + 2\langle d_{A_0+\alpha}u_0, d_{A_0+\alpha}v \rangle = \\ &= \|d_{A_0}v\|^2 + \|\alpha(v)\|^2 + 2\langle d_{A_0}v, \alpha(v) \rangle + 2\langle \alpha(u_0), d_{A_0}v \rangle + 2\langle \alpha(u_0), \alpha(v) \rangle. \end{aligned}$$

But

$$d_{A_0}^*(\alpha(u_0)) = - * d_{A_0}(*\alpha(u_0)) = - * d_{A_0}(*\alpha)(u_0) \pm *( *\alpha \wedge d_{A_0}u_0) = 0$$

(because  $d_{A_0}^*(\alpha) = 0$  and  $d_{A_0}u_0 = 0$ ), so the term  $2\langle \alpha(u_0), d_{A_0}v \rangle$  on the right vanishes. On the other hand it holds pointwise

$$\begin{aligned} (\alpha(u_0), \alpha(v)) &= ((u_0 \wedge_F a)(u_0), (u_0 \wedge_F a)(v)) = \\ &= (a, (u_0, v)a - (a, v)u_0) = |a|^2(u_0, v) = \frac{1}{2}|\alpha|^2(u_0, v), \\ 1 &= |u_0 + v|^2 = |u_0|^2, \quad 2(u_0, v) = -|v|^2. \end{aligned}$$

Therefore

$$\begin{aligned} \|d_{A_0+\alpha}u\|^2 - \|d_{A_0+\alpha}u_0\|^2 &= \|d_{A_0}v\|^2 + \|\alpha(v)\|^2 + 2\langle d_{A_0}v, \alpha(v) \rangle - \frac{1}{2} \int_M |\alpha|^2 |v|^2 \\ &\geq \|d_{A_0}v\|^2 - c_1 \sup |\alpha| \|d_{A_0}v\|_{L^2} \|v\|_{L^2} - c_2 \sup |\alpha|^2 \|v\|_{L^2}^2 \end{aligned}$$

Since the same computation also applies to  $-u_0$ , we get

$$(5) \quad E_{A_0+\alpha}(u) - E_{A_0+\alpha}(u_0) \geq \|d_{A_0}v'\|^2 - c_1 \sup |\alpha| \|d_{A_0}v'\| \|v'\| - c_2 \sup |\alpha|^2 \|v'\|^2$$

$$E_{A_0+\alpha}(u) - E_{A_0+\alpha}(u_0) \geq \|d_{A_0}v\|^2 - c_1 \sup |\alpha| \|d_{A_0}v\| \|v\| - c_2 \sup |\alpha|^2 \|v\|^2$$

Since  $\|u\|_{L^2}^2 = \|u_0\|_{L^2}^2 = \text{Vol}(M)$ , one has  $(u - u_0) \perp_{L^2} (u + u_0)$ , so the triangle  $(u_0, u, -u_0)$  is  $L^2$ -right at the vertex  $u$ , and one has

$$\sin^2 \angle(\mathbb{R}v, \mathbb{R}u_0) + \sin^2 \angle(\mathbb{R}v', \mathbb{R}u_0) = 1.$$

Therefore, either  $\sin \angle(\mathbb{R}v, \mathbb{R}u_0) \geq \frac{1}{\sqrt{2}}$  or  $\sin \angle(\mathbb{R}v', \mathbb{R}u_0) \geq \frac{1}{\sqrt{2}}$ . Suppose we are in the first case.

We get the inequality:

$$\|v\|_{L^2}^2 \leq 2 \|\text{pr}_{\mathbb{R}u_0^\perp} v\|^2 \leq \frac{2}{\lambda(A_0)} \|d_{A_0}(v)\|^2,$$

where  $\text{pr}_{\mathbb{R}u_0^\perp}$  stands for the  $L^2$ -orthogonal projection on the  $L^2$ -orthogonal complement of the line  $\mathbb{R}u_0 = \ker(d_{A_0})$  and  $\lambda(A_0)$  is the first positive eigenvalue of  $\Delta_{A_0} : A^0(F) \rightarrow A^0(F)$ . In other words,  $\frac{1}{\sqrt{\lambda(A_0)}}$  is the norm of the inverse

$G_{A_0} : \text{im}(d_{A_0}) \rightarrow \ker(d_{A_0})^\perp$  of  $d_{A_0}$  with respect to the  $L^2$ -norms.

Using (5) we get an estimate of the form

$$E_{A_0+\alpha}(u) - E_{A_0+\alpha}(u_0) \geq \|d_{A_0}v\|^2 - c_1(A_0) \sup |\alpha| \|d_{A_0}v\|^2 - c_2(A_0) \sup |\alpha|^2 \|d_{A_0}v\|^2$$

for positive constants  $c_1(A_0)$ ,  $c_2(A_0)$ . Taking

$$\varepsilon(A_0) = \min \left( \frac{1}{4c_1(A_0)}, \sqrt{\frac{1}{2c_2(A_0)}} \right),$$

we get for  $\sup |\alpha| \leq \varepsilon(A_0)$

$$E_{A_0+\alpha}(u) - E_{A_0+\alpha}(u_0) \geq \frac{1}{2} \|d_{A_0} v\|^2 \geq \frac{\lambda(A_0)}{4} \|v\|^2,$$

which is strictly positive unless  $v = 0$ , i.e.  $u = u_0$ . The same argument applies in the case  $\sin \angle(\mathbb{R}v', \mathbb{R}u_0) \geq \frac{1}{\sqrt{2}}$  by replacing  $u$  with  $-u_0$  and  $v$  with  $v'$ . ■

The inequality  $\|\alpha\|_{L^\infty} < \varepsilon(A_0)$  as  $A_0$  varies in  $\mathcal{A}'(F)$  defines a gauge invariant neighborhood  $\mathcal{N}$  of the zero section in the normal bundle  $N$  of this submanifold.

**Corollary 1.10.** *The restriction of the natural map*

$$\nu : N \longrightarrow \mathcal{A}(F), \quad \nu(A_0, \alpha) := A_0 + \alpha$$

*to  $\mathcal{N}$  is injective.*

**Proof:** Indeed, if  $\nu(A_0, \alpha) = \nu(B_0, \beta) = A$ , then one must have  $\ker(d_{A_0}) = \ker(d_{B_0})$ , because the absolute minimum of  $E_A$  on  $\Gamma(S(F))$  is unique up to sign. By Corollary 1.4, we obtain  $A_0 = B_0$ , so finally  $\alpha = \beta$ . ■

Combining with Lemma 1.7, we get

**Theorem 1.11.** *There exists a gauge invariant neighborhood  $\mathcal{U}$  of the zero section in the bundle  $N$  which (after suitable Sobolev completions) is mapped diffeomorphically onto its image via the natural map  $\nu$ .*

**Remark.** *The map  $\nu|_{\mathcal{U}}$  can be regarded as a system of "gauge equivariant polar coordinates" around the submanifold  $\mathcal{A}'(F)$ .*

**Remark.** *In the finite dimensional framework, one can prove easily that in general, for any submanifold  $X$  of a Riemannian manifold  $Y$  there exists a neighborhood of the zero section in the normal bundle  $N_{X/Y}$  which is mapped diffeomorphically via the exponential map onto a normal neighborhood of  $X$ . However, in the infinite dimensional framework, the problem is much more difficult. Theorem 1.11 solves this problem in the special case of the embedding  $\mathcal{A}'(F) \subset \mathcal{A}(F)$ .*

**1.3. Normal neighborhoods of the reduction loci in the space of Hermitian connections.** Let  $E$  be a rank 2 Hermitian bundle over a 4 manifold  $M$  and denote  $D := \det(E)$ ,  $d := c_1(D)$ . Consider the involution  $i_d : x \mapsto d - x$  on  $H^2(M, \mathbb{Z})$ . A congruence class  $\lambda \in H^2(M, \mathbb{Z}) / \langle i_d \rangle$  will be called a topological decomposition of  $E$  if it coincides with the set of Chern classes of the terms of a splitting of  $E$  as direct sum of line bundles (i.e. when  $x(d - x) = c_2(E)$  for  $x \in \lambda$ ).

Fix a connection  $a \in \mathcal{A}(D)$  and denote by  $\mathcal{A}_a(E)$  the affine space of connections on  $E$  inducing  $a$  on  $D$ . Our gauge group is the group  $\mathcal{G}_E := \Gamma(SU(E))$  of determinant 1 unitary isomorphisms of  $E$ . A connection will be called *simply reducible* if  $\dim(\ker(d_A : A^0(su(E)) \rightarrow A^1(su(E)))) = 1$ . Such a connection admits precisely two parallel line subbundles (which, of course, might be isomorphic), and these subbundles give an  $A$ -parallel orthogonal splitting of  $E$ . If  $A$  admits a parallel line bundle  $L$  with  $2c_1(L) \neq d$ , then  $A$  is automatically simply reducible and  $L$ ,  $L^\perp$  are the unique  $A$ -parallel line subbundles of  $E$ . In particular, if  $d \notin 2H^2(M, \mathbb{Z})$ ,

then any reducible connection on  $E$  is simply reducible. If we fix a line subbundle  $L \hookrightarrow E$ , there exists a natural bijection between the simply reducible connections on  $E$  for which  $L$  is parallel, and the subspace  $\mathcal{A}^*(L) \subset \mathcal{A}(L)$  of abelian connections  $b \in \mathcal{A}(L)$  for which  $b \not\cong a \otimes b^\vee$ . One has automatically  $\mathcal{A}^*(L) = \mathcal{A}(L)$  when  $2c_1(L) \neq d$ , whereas  $\mathcal{A}^*(L)$  is the complement of  $2^{b_1(N)}$  gauge orbits in  $\mathcal{A}(L)$  when  $2c_1(L) = d$ .

Let  $\lambda$  be a topological decomposition of  $E$  and denote by  $\mathcal{A}_a^\lambda(E)$  the subspace of *simple* reducible connections  $A \in \mathcal{A}_a(E)$  with the property that the set of the Chern classes of the two  $A$ -parallel line subbundles of  $E$  coincides with  $\lambda$ . Such a connection will be called  $\lambda$ -reducible. Denote also by  $\Gamma^\lambda(S(su(E)))$  the set of sections  $u \in \Gamma(S(su(E)))$  with the property that the set  $c(u)$  of Chern classes of the eigen line subbundles of  $u$  coincides with  $\lambda$ . Putting

$$(6) \quad L_u := \ker\left(u - \frac{i}{\sqrt{2}}\text{id}_E\right)$$

these eigen line subbundles are  $L_{\pm u}$ . One has a natural surjection

$$w : \mathcal{A}_a^\lambda(E) \longrightarrow \Gamma^\lambda(S(su(E))) / \{\pm 1\}$$

which associates to every connection  $A \in \mathcal{A}_a^\lambda(E)$  the unordered pair of sections of the sphere bundle  $S(su(E))$  which are  $A$ -parallel.

For sufficiently large Sobolev index  $k$ , the space  $\Gamma(S(su(E)))_{k+1}$  becomes a Banach manifold and the subset  $\Gamma^\lambda(S(su(E)))_{k+1}$  is open and closed in the space  $\Gamma(S(su(E)))_{k+1}$ , so it is a union of connected components. The gauge group  $\mathcal{G}_{E,k+1}$  acts smoothly on  $\Gamma(S(su(E)))_{k+1}$ , leaving invariant  $\Gamma^\lambda(S(su(E)))_{k+1}$ ; on  $\Gamma^\lambda(S(su(E)))_{k+1} / \{\pm 1\}$  this gauge group acts transitively. The same arguments as in the proof of Proposition 1.5 show that

**Proposition 1.12.** *The subset  $\mathcal{A}_a^\lambda(E)_k \subset \mathcal{A}_a(E)_k$  is a submanifold, which is a locally trivial affine bundle over  $\Gamma^\lambda(S(su(E)))_{k+1} / \{\pm 1\}$ . The fiber over a class  $[u] \in \Gamma(S(su(E)))_{k+1} / \{\pm 1\}$  can be naturally identified with the space of abelian connections  $\mathcal{A}^*(L_u)_k$ .*

Fix a reducible connection  $A \in \mathcal{A}_a^\lambda(E)$  with  $w(A) = [u]$ . Put  $S_u := (L_u^{\otimes 2} \otimes D^\vee)$ . The bundle  $su(E)$  splits as an orthogonal sum of  $A$ -parallel summands:

$$su(E) = (M \times \mathbb{R}u) \oplus S_u = so(u^\perp) \oplus so(u^\perp)^\perp,$$

and, as in section 1.2, we obtain the following  $L^2$ -orthogonal decomposition of the tangent space  $T_A(\mathcal{A}_a(E)) = A^1(su(E))$  at  $A$ :

$$A^1(su(E)) = A^1(M, \mathbb{R})u \oplus A^1(S_u) =$$

$$A^1(M, \mathbb{R})u \oplus d_A[A^0(S_u)] \oplus \ker [d_A^* : A^1(S_u) \rightarrow A^0(S_u)].$$

The first summand  $A^1(M, \mathbb{R})u$  is the tangent space of the fiber  $w^{-1}(u) \simeq \mathcal{A}^*(L_u)$  at  $A$ ,  $n_A := d_A[A^0(S_u)]$  is the normal space at  $A$  of this fiber in the submanifold  $\mathcal{A}_a^\lambda(E)$ , whereas the space  $N_A := \ker [d_A^* : A^1(S_u) \rightarrow A^0(S_u)]$  is the normal space at  $A$  of  $\mathcal{A}_a^\lambda(E)$  in the space of connections  $\mathcal{A}_a(E)$ .

We denote by  $N^\lambda \rightarrow \mathcal{A}_a^\lambda(E)$  the  $L^2$ -normal vector bundle of the submanifold  $\mathcal{A}_a^\lambda(E)$ , whose fiber over  $A \in \mathcal{A}_a^\lambda(E)$  is  $N_A$ .

Using Theorem 1.11 we obtain the following important result, which gives an  $L^2$ -normal neighborhood of the submanifold  $\mathcal{A}_a^\lambda(E)$  of simple reductions of type  $\lambda$ , and a system of polar coordinates around this submanifold.

**Theorem 1.13.** *There exists a gauge invariant neighborhood  $\mathcal{U}^\lambda$  of the zero section in the normal bundle  $N^\lambda \rightarrow \mathcal{A}_a^\lambda(E)$  which (after suitable Sobolev completions) is mapped diffeomorphically onto its image via the natural map*

$$\nu : N^\lambda \rightarrow \mathcal{A}_a(E) , \quad (A_0, \alpha) \mapsto A_0 + \alpha .$$

*The elements of the orthogonal slice  $\nu(N_{A_0}^\lambda \cap \mathcal{U}^\lambda) \subset \mathcal{A}_a(E)$  through  $A_0 \in \mathcal{A}_a^\lambda(E)$  are connections  $A$  for which the two elements of*

$$\ker(d_{A_0} : A^0(\mathfrak{su}(E)) \rightarrow A^1(\mathfrak{su}(E))) \cap \Gamma(S(\mathfrak{su}(E)))$$

*(which are  $A_0$ -parallel) are the unique energy-minimizing  $A$ -harmonic sections in  $\Gamma(S(\mathfrak{su}(E)))$ .*

From now on we will always assume that  $\mathcal{U}^\lambda$  is defined by an inequality of the form  $\|\alpha\|_{L^\infty} \leq \varepsilon(A_0)$ , where  $A_0 \mapsto \varepsilon(A)$  is continuous and gauge invariant (see Lemma 1.7 and Lemma 1.9).

**1.4. Twisted reductions.** When the base manifold has nontrivial first homology group  $H_1(M, \mathbb{Z})$ , one also has to take into account the twisted reductions, i.e. the connections which are irreducible but whose pull-back on a double cover of  $M$  become reducible. Although the stabilizer  $\mathcal{G}_{E,A}$  of such a connection is just the center  $\{\pm \text{id}_E\}$  of the gauge group, these loci of twisted connections and the geometry of the instanton moduli spaces around these loci must be studied in detail; the reason is simple: the classical transversality results with respect to metric variations [DK] fails not only at a reduction, but also at a twisted reduction, so it is not clear whether one can achieve regularity of an instanton moduli space at such a point by perturbing the metric.

Let  $\rho : \pi_1(M, x_0) \rightarrow \{\pm 1\}$  be a group epimorphism and denote by  $\pi_\rho : M_\rho \rightarrow M$  the double cover associated with  $\ker(\rho)$ . The tautological involution of  $M_\rho$  will be denoted by  $\iota$ . A connection  $A \in \mathcal{A}_a(E)$  will be called  $\rho$ -twisted reducible (or a  $\rho$ -twisted reduction) if it is irreducible, but its pullback  $\pi_\rho^*(A) \in \mathcal{A}_{\pi_\rho^*(a)}(\pi_\rho^*(E))$  is reducible. For such a connection one has an orthogonal  $\pi_\rho^*(A)$ -parallel splitting  $\pi_\rho^*(E) = L' \oplus L''$  and an isomorphism  $L'' \simeq \iota^*(L')$ .  $L'$  and  $L''$  are the unique  $\pi_\rho^*(A)$ -parallel subbundles of  $E$ , because, if not, the  $SO(3)$ -connection associated with  $\pi_\rho^*(A)$  would be trivial. In this case  $A$  will be projectively flat and the holonomy of  $A$  acts on the projective line  $\mathbb{P}(E_{x_0})$  by an involution. Therefore,  $A$  would admit (at least) two parallel line subbundles, contradicting the assumption that  $A$  was irreducible. In other words  $\pi_\rho^*(A)$  must be simply reducible of type  $\lambda = (l, \iota^*(l))$ , where  $l$  is a solution in  $H^2(M_\rho, \mathbb{Z})$  of the system

$$(7) \quad l + \iota^*(l) = \pi_\rho^*(d) , \quad l \cdot \iota^*(l) = 2c_2(E) .$$

For a  $\rho$ -twisted reduction  $A$  the  $SO(3)$ -bundle  $\mathfrak{su}(E)$  has an  $A$ -parallel splitting

$$\mathfrak{su}(E) = R_\rho \oplus F ,$$

where  $R_\rho$  is the Euclidean real line subbundle of  $\mathfrak{su}(E)$  consisting of trace-free anti-Hermitian endomorphisms  $u_x \in \mathfrak{su}(E_x)$  whose eigen lines are  $L'_{\tilde{x}}$ ,  $L''_{\tilde{x}} = L'_{\iota(\tilde{x})}$ , where  $\tilde{x}$  is a lift of  $x \in M$  in  $M_\rho$ , and  $L', L''$  are the  $\pi_\rho^*(A)$ -parallel line subbundles of  $\pi_\rho^*(E)$ .  $R_\rho$  is isomorphic with the non-orientable Euclidean real line bundle

associated with the representation  $\rho : \pi_1(M, x_0) \rightarrow O(1)$ . The second term  $F$  is an  $O(2)$ -bundle with  $\det(F) \simeq R_\rho$ , whose pull-back to  $M_\rho$  has two  $SO(2) = U(1)$ -reductions, isomorphic with  $[L']^\vee \otimes L''$  and  $[L'']^\vee \otimes L'$ .

We fix a topological decomposition  $\lambda = \{l, \iota^*(l)\}$  of  $\pi_\rho^*(E)$  and we denote by  $\mathcal{A}_a^\lambda(E)$  the subspace of  $\rho$ -twisted reductions  $A \in \mathcal{A}_a(E)$  with the property that  $\pi_\rho^*(A)$  is simply reducible of type  $\lambda$ . Denote by  $\Gamma_\iota(S(su(\pi_\rho^*(E))))$  the set of sections  $u$  of the sphere bundle of  $su(\pi_\rho^*(E))$  satisfying the property  $\iota^*(u) = -u$  and by  $\Gamma_\iota^\lambda(S(su(\pi_\rho^*(E))))$  the subset of  $\Gamma_\iota(S(su(\pi_\rho^*(E))))$  consisting of sections  $u$ , such that the Chern classes of the eigen line sub-bundles of  $\pi_\rho^*(u)$  are  $l, \iota^*(l)$ .

In the same way as in the case of non-twisted connections one gets a locally trivial, gauge equivariant fibration

$$w : \mathcal{A}_a^\lambda(E) \longrightarrow \Gamma_\iota^\lambda(S(su(\pi_\rho^*(E)))) / \{\pm 1\} ,$$

and the fiber over a class  $[u]$  can be identified with the subspace

$$\mathcal{A}_\iota^*(L_{\pi_\rho^*(u)}) \subset \mathcal{A}(L_{\pi_\rho^*(u)})$$

of abelian connections  $b \in \mathcal{A}(L_{\pi_\rho^*(u)})$  satisfying

$$b \otimes \iota^*(b) = \iota^*(a) \text{ (via the obvious isomorphism } L \otimes \iota^*(L) \simeq \pi_\rho^*(D)) , b \not\cong \iota^*(b).$$

The second condition is superfluous when  $l \neq \iota^*(l)$ . The space  $A^1(su(E))$  splits as

$$A^1(su(E)) = A^1(R_\rho) \oplus d_A[A^0(F)] \oplus \ker [d_A^* : A^1(F) \rightarrow A^0(F)] ,$$

where the third term can be identified with the normal space  $N_A^\lambda$  at  $A$  of the submanifold  $\mathcal{A}_a^\lambda(E)$  of  $\rho$ -twisted connections of type  $\lambda$ .

Using similar methods as in the proofs of Theorems 1.11, 1.13 one gets easily the following existence theorem for  $L^2$ -normal neighborhoods of the loci of twisted reductions.

**Theorem 1.14.** *There exists a gauge invariant neighborhood  $\mathcal{U}^\lambda$  of the zero section in the normal bundle  $N^\lambda \rightarrow \mathcal{A}_a^\lambda(E)$  which (after suitable Sobolev completions) is mapped diffeomorphically onto its image via the natural map*

$$\nu : N^\lambda \rightarrow \mathcal{A}_a(E) , (A_0, \alpha) \mapsto A_0 + \alpha .$$

*The elements of the orthogonal slice  $\nu(N_{A_0} \cap \mathcal{U}^\lambda) \subset \mathcal{A}_a(E)$  through  $A_0 \in \mathcal{A}_a^\lambda(E)$  are connections  $A \in \mathcal{A}_a(E)$  for which the two elements of the intersection*

$$\ker(d_{\pi_\rho^*(A_0)} : A^0(su(\pi_\rho^*(E))) \rightarrow A^1(su(\pi_\rho^*(E)))) \cap \Gamma_\rho(S(su(\pi_\rho^*(E))))$$

*(which are  $\pi_\rho^*(A_0)$ -parallel) are the unique energy-minimizing  $\pi_\rho^*(A)$ -harmonic sections in  $\Gamma(S(su(\pi_\rho^*(E))))$ .*

As in the non-twisted case we will suppose that  $\mathcal{U}^\lambda$  is defined by an inequality of the form  $\|\alpha\|_{L^\infty} \leq \varepsilon(A)$ , where the assignment  $\mathcal{A}_a^\lambda(E) \ni A \mapsto \varepsilon(A)$  is gauge invariant and continuous.

## 2. THE DONALDSON $\mu$ -CLASSES AROUND THE REDUCTIONS AND THE HOMOTOPY TYPE OF $\mathcal{B}_a(E)$

**2.1. The universal bundle around reductions.** The structure of the universal bundle around a single reduction is well-known. A complete description can be found in [DK] p. 186-187. However, for our purposes, this classical result is not

sufficient, because we will need the structure of the universal bundle around positive dimensional subspaces of reductions.

We recall that the universal  $SO(3)$ -bundle on  $\mathcal{B}_a^*(E) \times M$  is defined as

$$\mathbb{F} := \mathcal{A}_a^*(E) \times su(E) / \bar{\mathcal{G}}_E ,$$

where  $\bar{\mathcal{G}}_E := \mathcal{G}_E / \{\pm 1\}$  acts in the natural way on both factors. Alternatively, one can let  $\bar{\mathcal{G}}_E$  act from the right and define  $\mathbb{F}$  to be the bundle with fiber  $su(E)$  over  $\mathcal{B}_a^*(E)$  which is associated with the principal  $\bar{\mathcal{G}}_E$ -bundle  $\mathcal{A}_a^*(E) \rightarrow \mathcal{B}_a^*(E)$ . Let  $\lambda = \{l, d-l\}$  be a topological decomposition of  $E$ . We will assume for simplicity that  $2l \neq d$  which assures that  $\mathcal{A}(L) = \mathcal{A}^*(L)$  for every Hermitian line bundle of Chern class  $l$ ; in particular the fiber of the fibration  $w : \mathcal{A}_a^\lambda(E) \rightarrow \Gamma^\lambda(S(su(E))) / \{\pm 1\}$  over  $\{\pm u\}$  can be identified with the affine space  $\mathcal{A}(L_u)$  (see section 1.3). We will identify  $\mathcal{A}(L_u)$  with its image in  $\mathcal{A}_a(E)$ .

We will omit the upper script  $\lambda$  in the notations  $N^\lambda, \mathcal{U}^\lambda$  introduced in the previous section and we denote by  $N^*, \mathcal{U}^*$  the complement of the zero section in  $N$  and  $\mathcal{U}$  (see Theorem 1.13). Replacing  $\mathcal{U}$  by a smaller gauge invariant neighborhood if necessary, we may assume that  $\nu(\mathcal{U}^*) \subset \mathcal{A}_a^*(E)$ . We put

$$\mathcal{V} := \nu(\mathcal{U}) / \bar{\mathcal{G}}_E \subset \mathcal{B}_a(E) , \quad \mathcal{V}^* := \nu(\mathcal{U}^*) / \bar{\mathcal{G}}_E \subset \mathcal{B}_a^*(E) .$$

$\mathcal{V}$  is a neighborhood of the moduli space of  $\lambda$ -reductions  $\mathcal{B}_a^\lambda(E) := \mathcal{A}_a^\lambda(E) / \mathcal{G}_E$ . Fix a section  $u \in \Gamma^\lambda(S(su(E)))$  and denote by  $\mathcal{G}_u$  the subgroup of  $\mathcal{G}_E$  consisting of elements  $g \in \mathcal{G}$  which leave  $L_u$  invariant. We have a natural isomorphism  $\mathcal{G}_u \simeq \mathcal{G}$ , where  $\mathcal{G} := \mathcal{C}^\infty(M, S^1)$ . Put

$$\bar{\mathcal{G}} := \mathcal{G} / \{\pm 1\} ; \quad \bar{\mathcal{G}}_u := \mathcal{G}_u / \{\pm 1\} \subset \bar{\mathcal{G}}_E ,$$

and denote by  $\mathcal{G}^{x_0}, \mathcal{G}_E^{x_0}, \mathcal{G}_u^{x_0}$  the subgroups of  $\mathcal{G}$  (respectively  $\mathcal{G}_E, \mathcal{G}_u$ ) of elements  $f$  with  $f(x_0) = 1$ . Note that these subgroups are mapped injectively into  $\bar{\mathcal{G}}, \bar{\mathcal{G}}_E, \bar{\mathcal{G}}_u$ , so we will use the same notations for the corresponding subgroups of these groups.

The main point which will be used in our computation is that the fixing of the section  $u$  defines a  $\bar{\mathcal{G}}_u$ -reduction of the restriction of the principal  $\bar{\mathcal{G}}_E$ -bundle  $\mathcal{A}_a^*(E) \rightarrow \mathcal{B}_a^*(E)$  to the subspace  $\mathcal{V}^* \subset \mathcal{B}_a^*(E)$ .

**Proposition 2.1.** *Suppose  $\lambda = \{l, d-l\}$  with  $2l \neq d$ . Let  $u \in \Gamma^\lambda(S(su(E)))$  and let  $N_u \subset \mathcal{A}(L_u) \times A^1(S_u)$  be the restriction of the normal bundle  $N$  to the fiber  $w^{-1}([u]) \simeq \mathcal{A}(L_u)$ . Then*

- (1)  $\nu(N_u \cap \mathcal{U})$  is the submanifold of the normal neighborhood  $\nu(\mathcal{U})$  consisting of connections for which  $\pm u$  are harmonic and energy minimizing.
- (2) The embedding  $\nu|_{N_u \cap \mathcal{U}} : N_u \cap \mathcal{U} \hookrightarrow \mathcal{A}_a(E)$  induces isomorphisms

$$\mathcal{V} \simeq N_u \cap \mathcal{U} / \bar{\mathcal{G}}_u , \quad \mathcal{V}^* \simeq N_u^* \cap \mathcal{U} / \bar{\mathcal{G}}_u .$$

- (3) The map  $\nu$  induces an isomorphism between  $\mathcal{V}$  and the cone bundle over the projectivization  $\mathbb{P}(\bar{N}_u)$  of the vector bundle

$$\bar{N}_u := N_u / \mathcal{G}_u^{x_0}$$

over  $\mathcal{B}(L_u) = \mathcal{A}(L_u)/\mathcal{G}_u^{x_0}$ , and  $\mathcal{V}^*$  is identified with the complement of the vertex section in this cone bundle. In particular one has a homotopy equivalence

$$(8) \quad \mathcal{V}^* \xrightarrow{h} \mathbb{P}(\bar{N}_u) \subset \mathcal{A}(L_u) \times_{\mathcal{G}_u^{x_0}} \mathbb{P}(A^1(S_u)) .$$

(4)  $\nu(N_u \cap \mathcal{U}^*)$  is a  $\bar{\mathcal{G}}_u$ -reduction of the restriction of the principal  $\bar{\mathcal{G}}_E$ -bundle  $\mathcal{A}_a^*(E) \rightarrow \mathcal{B}_a^*(E)$  to  $\mathcal{V}^*$ .

**Proof:** The first statement follows easily from Theorem 1.13. The second and the third statements are obvious. For the fourth, it suffices to prove that the  $\bar{\mathcal{G}}_E$ -orbit of a point  $\nu(t) \in \nu(\mathcal{U}^*)$  intersects  $\nu(N_u \cap \mathcal{U}^*)$  along a  $\bar{\mathcal{G}}_u$ -orbit; since  $\nu$  is injective on  $\mathcal{U}$ , it suffices to show that the  $\bar{\mathcal{G}}_E$ -orbit of  $t = (b, \alpha) \in N^*$  intersects  $N_u^*$  along a  $\bar{\mathcal{G}}_u$ -orbit. Since  $\bar{\mathcal{G}}_E$  acts transitively on  $\Gamma^\lambda(S(su(E)))/\{\pm 1\}$ , we see that  $t \cdot \bar{\mathcal{G}}_E \cap N_u^* \neq \emptyset$ , so we can suppose  $t \in N_u^*$ . If an element  $g \in \mathcal{G}_E$  maps  $t$  into  $N_u^*$ , then  $g(b) \in \mathcal{A}(L_u) \subset \mathcal{A}_a(E)$ , i.e.  $u$  is both  $b$ -parallel and  $g(b)$ -parallel. This implies  $\text{ad}_g(u) = \pm u$ . Since we assumed that  $l \neq d - l$ , we have  $L_u \not\cong L_{-u}$ , so necessarily  $\text{ad}_g(u) = u$  i.e.  $g \in \mathcal{G}_u$ .  $\blacksquare$

**Corollary 2.2.** *The open subspace  $\mathcal{V}^* \subset \mathcal{B}_a^*(E)$  has the homotopy type of the product  $[S^1]^{b_1(M)} \times \mathbb{P}^\infty$ .*

**Proof:** Let  $g$  be a Riemannian metric on  $M$ . The moduli space  $\mathcal{M}_g^{YM}(L_u)$  of Yang-Mills connections on  $L_u$  with respect to  $g$  is isomorphic to

$$iH^1(X, \mathbb{R})/2\pi iH^1(M, \mathbb{Z}) \simeq [S^1]^{b_1(M)}$$

and the inclusion  $\mathcal{M}_g^{YM}(L_u) \hookrightarrow \mathcal{B}(L_u)$  is a homotopy equivalence. On the other hand, since  $\mathcal{M}_g^{YM}(L_u)$  is compact, the restriction of the infinite rank vector bundle  $\bar{N}_u$  to this subspace is trivial. This completes the proof.  $\blacksquare$

The decomposition

$$su(E) = [M \times (i\mathbb{R})] \oplus S_u$$

is  $\mathcal{G}_u$ -invariant. Therefore

**Corollary 2.3.** *The restriction  $\mathbb{F}|_{\mathcal{V}^* \times M}$  decomposes as a direct sum*

$$\mathbb{F}|_{\mathcal{V}^* \times M} \simeq [(\mathcal{V}^* \times M) \times i\mathbb{R}] \oplus \mathcal{S}_u ,$$

where  $\mathcal{S}_u$  is the complex line bundle on  $\mathcal{V}^* \times M$  defined by

$$\mathcal{S}_u := \nu(N_u \cap \mathcal{U}^*) \times S_u / \bar{\mathcal{G}}_u .$$

In particular  $p_1(\mathbb{F}) = c_1(\mathcal{S}_u)^2$ .

The cohomology algebra of  $\mathcal{V}^*$  can be easily described explicitly using Corollary 2.2. The construction below yields generators with explicit geometric interpretation.

Put  $L = L_u$ ,  $S = S_u = L^{\otimes 2} \otimes D^\vee$ ,  $\mathcal{S} = \mathcal{S}_u$  to save on notations. The subbundle  $N_u \subset \mathcal{A}(L) \times A^1(S)$  has an intrinsic interpretation in terms of  $L$ : it is just the bundle  $\mathcal{K} = \ker \delta$  of kernels of the family of operators

$$\delta := (d_{b^{\otimes 2} \otimes a^\vee})_{b \in \mathcal{A}(L)}$$

and  $\mathcal{U}_u := \mathcal{U} \cap N_u$  is a  $\bar{\mathcal{G}}$ -invariant neighborhood  $\mathcal{U}_u$  of the zero section in this bundle. The bundle  $\mathcal{K}$  descends to a bundle  $\bar{\mathcal{K}} := \mathcal{K}/\mathcal{G}^{x_0}$  over  $\mathcal{B}(L)$ . Using the isomorphism  $\mathcal{V}^* \simeq \mathcal{U}_u^*/\bar{\mathcal{G}}$  induced by  $\nu$ , one can identify  $\mathcal{S}$  with the line bundle

$$\mathcal{U}_u^* \times S/\bar{\mathcal{G}} \longrightarrow \mathcal{U}_u^*/\bar{\mathcal{G}} \times M \subset \mathcal{K}^*/\bar{\mathcal{G}} \times M = \bar{\mathcal{K}}^*/S^1 \times M .$$

Therefore,  $\mathcal{S}$  can be identified (via  $\nu$ ) to the restriction to  $[\mathcal{U}_u^*/\bar{\mathcal{G}}] \times M$  of the line bundle

$$\mathbb{S} := \mathcal{A}(L) \times (A^1(S) \setminus \{0\}) \times S/\bar{\mathcal{G}}$$

on  $\mathcal{B}_u^*(E) \times M$ , where  $\mathcal{B}_u^*(E)$  is the quotient

$$\mathcal{B}_u^*(E) := \mathcal{A}(L) \times (A^1(S) \setminus \{0\})/\bar{\mathcal{G}} = \mathcal{A}(L) \times_{\mathcal{G}^{x_0}} [\mathbb{R}_{>0} \times \mathbb{P}(A^1(S))] ,$$

which can be regarded as a locally trivial bundle over  $\mathcal{B}(L)$  with fiber  $\mathbb{R}_{>0} \times \mathbb{P}(A^1(S))$ . The space  $\mathcal{B}_u^*(E)$  is very much similar to the infinite dimensional gauge quotient of the space of irreducible configurations in Seiberg-Witten theory. More precisely, let in general  $V$  be a line bundle and  $W$  a complex vector bundle on  $M$ . The natural map

$$p_{V,W} : \mathcal{B}^*(V, W) := \mathcal{A}(V) \times [A^0(W) \setminus \{0\}]/\mathcal{G} \longrightarrow \mathcal{B}(V)$$

is a locally trivial bundle over  $\mathcal{B}(V)$  with fiber  $\mathbb{R}_{>0} \times \mathbb{P}(A^0(W))$ . On the product  $\mathcal{B}(V) \times M$  one has a *tautological line bundle*  $\mathbb{V}_W$  defined by

$$\mathbb{V}_W := \mathcal{A}(V) \times [A^0(W) \setminus \{0\}] \times V/\mathcal{G} .$$

**Lemma 2.4.** *With the notations and assumptions above one has*

- (1) *There exists a natural isomorphism*

$$\delta_V : H_1(M, \mathbb{Z}) \rightarrow H^1(\mathcal{B}(V))$$

*which induces an isomorphism  $H^*(\mathcal{B}^*(V), \mathbb{Z}) \simeq \Lambda^*(H_1(M, \mathbb{Z}))$ .*

- (2) *The morphism  $p_{V,W}^* \circ \delta_V$  induces an isomorphism*

$$H^*(\mathcal{B}^*(V, W), \mathbb{Z}) \simeq \Lambda^*(H_1(M, \mathbb{Z})) \otimes \mathbb{Z}[h_{VW}] ,$$

*where  $h_{VW}$  is a degree 2-cohomology class defined as the Chern class of the principal  $S^1$ -bundle*

$$\mathcal{A}(V) \times [A^0(W) \setminus \{0\}]/\mathcal{G}^{x_0} \longrightarrow \mathcal{A}(V) \times [A^0(W) \setminus \{0\}]/\mathcal{G} = \mathcal{B}^*(V, W) .$$

*This class restricts to the canonical (tautological) class of the fibers of the map  $p_{V,W} : \mathcal{B}^*(V, W) \rightarrow \mathcal{B}(V)$  (which are homotopically equivalent to  $\mathbb{P}(A^0(W))$ ).*

- (3) *The Chern class of the tautological line bundle  $\mathbb{V}_W$  on  $\mathcal{B}^*(V, W) \times M$  is*

$$c_1(\mathbb{V}_W) = p_1^*(h_{VW}) \otimes 1 + \delta_V + 1 \otimes p_2^*(c_1(V)) .$$

*Here we denoted by the same symbol  $\delta_V$  the element in  $H^1(\mathcal{B}(V)) \otimes H^1(M, \mathbb{Z})$  defined by this morphism, as well as the pullback of this element via the projection  $\mathcal{B}^*(V, W) \times M \rightarrow \mathcal{B}(V) \times M$ .*

**Proof:** The proof uses the same arguments which are used for the computation of the cohomology algebra of the moduli space of irreducible configurations in Seiberg-Witten theory (see for instance [OT1]). ■



**Corollary 2.5.** *The line bundle  $\mathbb{S}$  can be identified with the pull-back of  $\mathbb{S}_{\Lambda^1 \otimes S}$  via the composition*

$$\mathcal{B}_u^*(E) \longrightarrow \mathcal{B}^*(S, \Lambda^1 \otimes S)$$

induced by

$$\mathcal{A}(L) \times [A^1(S) \setminus \{0\}] \ni (b, \alpha) \mapsto (b^{\otimes 2} \otimes a^\vee, \alpha) \in \mathcal{A}(S) \times [A^1(S) \setminus \{0\}] .$$

In particular one has

$$c_1(\mathbb{S}) = p_1^*(h_u) \otimes 1 + 2\delta_L + p_2^*(c_1(S)) ,$$

where  $h_u$  is the Chern class of the principal  $\bar{\mathcal{G}}/\mathcal{G}^{x_0} = S^1$ -bundle

$$\mathcal{A}(L) \times (A^1(S) \setminus \{0\})/\mathcal{G}^{x_0} \longrightarrow \mathcal{A}(L) \times (A^1(S) \setminus \{0\})/\bar{\mathcal{G}} = \mathcal{B}_u^*(E) .$$

**Proof:** The map  $(b, \alpha) \mapsto (b^{\otimes 2} \otimes a^\vee, \alpha)$  induces an isomorphism  $u$  which fits in the diagram

$$\begin{array}{ccc} \mathcal{B}_u^*(E) = \mathcal{A}(L) \times [A^0(\Lambda^1 \otimes S) \setminus \{0\}]/\bar{\mathcal{G}} & \xrightarrow{u} & \mathcal{A}(S) \times [A^0(\Lambda^1 \otimes S) \setminus \{0\}]/\mathcal{G}^2 \xrightarrow{v} \\ & & \xrightarrow{v} \mathcal{A}(S) \times [A^0(\Lambda^1 \otimes S) \setminus \{0\}]/\mathcal{G} = \mathcal{B}^*(S, \Lambda^1 \otimes S) , \end{array}$$

where  $v$  is the obvious epimorphism. The map

$\mathcal{A}(L) \times [A^1(S) \setminus \{0\}] \times S \ni (b, \alpha, s) \mapsto (b^{\otimes 2} \otimes a^\vee, \alpha, s) \in \mathcal{A}(S) \times [A^1(S) \setminus \{0\}] \times S$  induces an isomorphism  $\mathbb{S} \simeq (v \circ u \times \text{id}_M)^*(\mathbb{S}_{\Lambda^1 \otimes S})$ . It suffices to notice that

$$(v \circ u)^*(h_{S, \Lambda^1 \otimes S}) = h_u , \quad (v \circ u)^*(\delta_S) = 2\delta_L .$$

■

**Corollary 2.6.** *Suppose that  $b_+(M) = 0$ . The restrictions of the Donaldson  $\mu$ -classes to  $\mathcal{V}^*$  are given by the formulae*

$$\begin{aligned} \mu(\lambda) &= -\langle \delta_L \cup_M c_1(S), \lambda \rangle \in H^1(\mathcal{V}^*, \mathbb{Z}) \quad \forall \lambda \in H_3(M, \mathbb{Z}) , \\ \mu(x) &= -\frac{1}{2} \langle c_1(S), x \rangle h_u \in H^2(\mathcal{V}^*, \mathbb{Q}) \quad \forall x \in H_2(X, \mathbb{Z}) , \\ \mu(\sigma) &= -\delta_L(\sigma) \cup h_u \in H^3(\mathcal{V}^*, \mathbb{Z}) \quad \forall \sigma \in H_1(X, \mathbb{Z}) , \\ \nu := \mu(*) &= -\frac{1}{4} h_u^2 \in H^4(\mathcal{V}^*, \mathbb{Q}) . \end{aligned}$$

**Proof:** Every monomial of the form  $a \cup b$ , with  $a, b \in H^1(B, \mathbb{Z})$  satisfies the relation  $(a \cup b)^2 = 0$  so, since we supposed  $b_+(M) = 0$ , one gets easily that  $a \cup b \in \text{Tors}(H^2(M, \mathbb{Z}))$ . Therefore

$$\text{im}(\cup : H^1(M, \mathbb{Z}) \otimes H^1(M, \mathbb{Z}) \rightarrow H^2(M, \mathbb{Z})) \subset \text{Tors}(H^2(M, \mathbb{Z})) .$$

This implies  $\delta_L^2 = 0$ . Therefore

$$\begin{aligned} -\frac{1}{4} p_1(\mathbb{F}|_{\mathcal{V}^*}) &= -\frac{1}{4} c_1(\mathbb{S})^2 = \\ &= -\frac{1}{4} [p_1^*(h_u^2) \otimes 1 + 4p_1^*(h_u) \cup \delta_L + 2p_1^*(h_u) \cup p_2^*(c_1(S)) + 4\delta_L \cup p_2^*(c_1(S)) + p_2^*(c_1(S)^2)] . \end{aligned}$$

Using Donaldson's formula  $\mu(x) = -\frac{1}{4} p_1(\mathbb{F})/x$  for  $x \in H_*(X, \mathbb{Z})$ , one gets easily the claimed formulae. ■

**2.2. The topology of the moduli space  $\mathcal{B}_a(E)$ .** Describing the weak homotopy type of the moduli space  $\mathcal{B}_a^*(E)$  of *irreducible* connections is a well-known classical problem in gauge theory. This problem is treated in detail in [DK], where the authors also compute the rational cohomology algebra of this space. Surprisingly, describing the weak homotopy type of the whole moduli space  $\mathcal{B}_a(E)$  of connections is a delicate problem, which, to our knowledge, cannot be solved with similar methods.

Our result concerning the existence of  $L^2$ -normal neighborhoods of the reduction loci, gives a solution to this problem. Suppose for simplicity that  $d := c_1(D)$  is not divisible by 2 in  $H^2(M, \mathbb{Z})$ , so that any reducible connection  $A \in \mathcal{A}_a(L)$  is simply reducible. Let  $\Lambda_E$  be the set topological decomposition of  $E$ , i.e. the set of unordered pairs  $\lambda = \{l, d-l\}$  with  $l(d-l) = c_2(E)$ .

Using the notations and conventions of section 2.1, we get the following decomposition of the space  $\mathcal{B}_a(E)$  as a fibered sum:

$$(9) \quad \begin{array}{ccc} & \mathcal{B}_a(E) & \\ & \nearrow & \nwarrow \\ \mathcal{B}_a^*(E) & & [\coprod_{\lambda \in \Lambda_E} \mathcal{V}^\lambda] \\ & \nwarrow & \nearrow \\ & [\coprod_{\lambda \in \Lambda_E} \mathcal{V}^{\lambda^*}] & \end{array}$$

where  $\mathcal{V}^\lambda := \mathcal{U}^\lambda / \mathcal{G}_E$  is the gauge quotient of the gauge invariant  $L^2$ -normal neighborhood  $\mathcal{U}^\lambda$  of the reduction locus  $\mathcal{A}_a^\lambda(E)$  constructed in section 1.3. In this diagram the homotopy types of the terms are known (see Corollary 2.2). Choosing a Hermitian line bundle  $L$  with  $c_1(L) \in \lambda$ , one has

$$\mathcal{V}^\lambda \stackrel{h}{\simeq} \mathcal{B}(L) \stackrel{h}{\simeq} [S^1]^{b_1(M)} \quad , \quad \mathcal{V}^{\lambda^*} \stackrel{h}{\simeq} [S^1]^{b_1(M)} \times \mathbb{P}^\infty \quad ,$$

this description determines the homotopy type of the space  $\mathcal{B}_a(E)$ . In particular one can compute the cohomology of this space using the Mayer-Vietoris exact sequence.

**Example 1.** Suppose that  $H_1(M, \mathbb{Z}) \simeq \mathbb{Z}$ ,  $b_2(M) = b_2^-(M) = 2$ . The intersection form of  $M$  is diagonalizable over  $\mathbb{Z}$  (since  $b_2(M) = 2$ , this follows from the classification of unimodular symmetric bilinear forms, but can also be obtained as a special case of Donaldson's first theorem). Therefore, one can find a basis  $(e_1, e_2)$  in  $H^2(M, \mathbb{Z})$  such that  $e_i^2 = -1$ ,  $e_1 \cup e_2 = 0$ . It follows that  $d := e_1 + e_2$  is an integral lift of the Stiefel-Whitney class  $w_2(M)$ . Let now  $E$  be a rank 2-Hermitian bundle with  $c_1(E) = d$ ,  $c_2(E) = 0$  and put again  $D = \det(E)$ .

Our problem is to compute the degree  $k$ -cohomology of the space  $\mathcal{B}_a(E)$ , for  $1 \leq k \leq 4$  using the Mayer-Vietoris sequence applied to the decomposition (9). The set  $\Lambda_E$  has two elements:

$$\lambda_0 := \{0, d\} \quad , \quad \lambda_1 := \{e_1, e_2\} \quad .$$

Denote by  $\mathcal{V}_0, \mathcal{V}_1, \mathcal{V}_0^*, \mathcal{V}_1^*$  the corresponding subspaces of  $\mathcal{B}_a(E)$ , and put  $\mathcal{V} := \mathcal{V}_0 \cup \mathcal{V}_1, \mathcal{V}^* := \mathcal{V}_0^* \cup \mathcal{V}_1^*$ . We get exact sequences:

$$\begin{aligned} \dots &\longrightarrow H^{i-1}(\mathcal{B}_a^*(E)) \oplus H^{i-1}(\mathcal{V}) \longrightarrow H^{i-1}(\mathcal{V}^*) \longrightarrow H^i(\mathcal{B}_a(E)) \longrightarrow \\ &\longrightarrow H^i(\mathcal{B}_a^*(E)) \oplus H^i(\mathcal{V}) \longrightarrow H^i(\mathcal{V}^*) \longrightarrow \dots \end{aligned}$$

Using the standard description of the cohomology of  $\mathcal{B}_a^*(E)$  (see [DK]), we obtain

$$H^1(\mathcal{B}_a^*(E)) = \langle \mu(\lambda_0) \rangle_{\mathbb{Q}} \simeq \mathbb{Q}, \quad H^2(\mathcal{B}_a^*(E)) = \langle \mu(f_1), \mu(f_2) \rangle_{\mathbb{Q}} \simeq \mathbb{Q}^2,$$

where  $\lambda_0$  is a generator of  $H_3(M, \mathbb{Z})$  and  $f_i := \text{PD}(e_i)$ . Since we assumed  $b_1(X) = 1$ , we obtain easily using Poincaré duality that

$$\text{im}(\cup : H^1(M) \otimes H^2(M) \rightarrow H^3(M)) = 0$$

hence, by Corollary 2.6, we obtain  $\mu(\lambda_0)|_{\mathcal{V}^*} = 0$ . Note also that the restriction morphism  $H^i(\mathcal{V}) \rightarrow H^i(\mathcal{V}^*)$  is an isomorphism for  $i = 0, 1$  and is injective for all  $i$ . The exact sequence above for  $i = 1$  yields

$$H^1(\mathcal{B}_a(E)) = \ker(H^1(\mathcal{B}_a^*(E)) \oplus H^1(\mathcal{V}) \rightarrow H^1(\mathcal{V}^*)) = \langle (\mu(\lambda_0), 0) \rangle_{\mathbb{Q}} \simeq \mathbb{Q}.$$

Therefore, the cohomology class defined by the Chern-Simons functional [DK] associated with a hypersurface representing  $\lambda_0$  extends to the whole moduli space  $\mathcal{B}_a(E)$ . This is a general phenomenon. For  $i = 2$ , we obtain

$$H^2(\mathcal{B}_a(E)) = \ker(H^2(\mathcal{B}_a^*(E)) \rightarrow H^2(\mathcal{V}^*)).$$

On the other hand by Corollary 2.6

$$\mu(f_i)|_{\mathcal{V}_0^*} = \frac{1}{2} \langle e_1 + e_2, f_i \rangle h_{u_0}, \quad \mu(f_i)|_{\mathcal{V}_1^*} = \frac{1}{2} \langle e_2 - e_1, f_i \rangle h_{u_1},$$

where  $u_i \in \Gamma(S(\text{su}(E)))$  are chosen such that  $c_1(L_{u_0}) = 0, c_1(L_{u_1}) = e_1$  (so  $L_{u_i}$  is a line bundle representing the topological decomposition  $\lambda_i$ ). Therefore  $H^2(\mathcal{B}_a(E), \mathbb{Q}) = 0$ . Since the restriction morphism  $H^2(\mathcal{B}_a^*(E)) \rightarrow H^2(\mathcal{V}^*)$  is surjective and  $H^3(\mathcal{V}) = 0$ , the same exact sequence for  $i = 3$  yields:

$$H^3(\mathcal{B}_a(E)) = \ker [H^3(\mathcal{B}_a^*(E)) \rightarrow H^3(\mathcal{V}^*) = H^3(\mathcal{V}_0^*) \oplus H^3(\mathcal{V}_1^*)].$$

The space  $H^3(\mathcal{V}_i^*)$  is generated by  $\delta_{L_i}(\sigma_0) \cup h_{u_i}$ , where  $\sigma_0$  is a generator of  $H_1(M, \mathbb{Z})$ , whereas

$$H^3(\mathcal{B}_a^*(E)) = \langle \mu(\sigma_0), \mu(\lambda_0)\mu(f_1), \mu(\lambda_0)\mu(f_2) \rangle_{\mathbb{Q}}.$$

By Corollary 2.6 one has  $\mu(\sigma_0)|_{\mathcal{V}_i^*} = -\delta_{L_i}(\sigma_0)h_{u_i}$  and we have seen that  $\mu(\lambda_0)|_{\mathcal{V}^*} = 0$ . This shows that

$$H^3(\mathcal{B}_a(E)) = \langle \mu(\lambda_0)\mu(f_1), \mu(\lambda_0)\mu(f_2) \rangle_{\mathbb{Q}} \simeq \mathbb{Q}^2.$$

The cohomology space  $H^4(\mathcal{B}_a(E))$  fits in the exact sequence

$$0 \rightarrow \frac{H^3(\mathcal{V}_0^*) \oplus H^3(\mathcal{V}_1^*)}{\langle (\delta_{L_0}(\sigma_0)h_{u_0}, \delta_{L_1}(\sigma_0)h_{u_1}) \rangle} \rightarrow H^4(\mathcal{B}_a(E)) \rightarrow H^4(\mathcal{B}_a^*(E)) \rightarrow H^4(\mathcal{V}^*)$$

The space  $H^4(\mathcal{B}_a^*(E))$  is freely generated by the five classes

$$\mu(*), \mu(f_1)^2, \mu(f_2)^2, \mu(f_1)\mu(f_2), \mu(\lambda_0)\mu(\sigma_0),$$

whereas  $H^4(\mathcal{V}^*)$  is freely generated by  $h_{u_0}^2$  and  $h_{u_1}^2$ . This shows that

$$\ker(H^4(\mathcal{B}_a^*(E)) \rightarrow H^4(\mathcal{V}^*)) = \langle \mu(*) + \mu(f_1)^2, \mu(*) + \mu(f_2)^2, \mu(\lambda_0)\mu(\sigma_0) \rangle_{\mathbb{Q}} \simeq \mathbb{Q}^3.$$

The quotient on the left in the exact sequence above is 1-dimensional, so one obtains  $H^4(\mathcal{B}_a(E)) \simeq \mathbb{Q}^4$ .

**Remark.** A similar method can be used to compute the cohomology of the pair  $(\mathcal{B}_a^*(E), \mathcal{V}^*)$ , where  $\mathcal{V} = \coprod_{[L] \in R_E} \mathcal{V}^L$  is a normal neighborhood of the reducible locus in  $\mathcal{B}_a(E)$ .

**Example 2.** Let  $M$  be a 4-manifold with the topological properties considered in the example above. The exact sequence of the pair  $(\mathcal{B}_a^*(E), \mathcal{V}^*)$

$$\rightarrow H^{i-1}(\mathcal{B}_a^*(E)) \rightarrow H^{i-1}(\mathcal{V}^*) \rightarrow H^i(\mathcal{B}_a^*(E), \mathcal{V}^*) \rightarrow H^i(\mathcal{B}_a^*(E)) \rightarrow H^i(\mathcal{V}^*) \rightarrow$$

written for  $i = 4$  shows that the natural morphism  $H^4(\mathcal{B}_a^*(E), \mathcal{V}^*) \rightarrow H^4(\mathcal{B}_a(E))$  (induced by the restriction morphism  $H^*(\mathcal{B}_a(E), \mathcal{V}) \rightarrow H^*(\mathcal{B}_a(E))$  and the excision isomorphism  $H^*(\mathcal{B}_a(E), \mathcal{V}) \rightarrow H^*(\mathcal{B}_a^*(E), \mathcal{V}^*)$ ) is an isomorphism.

### 3. THE INSTANTON MODULI SPACE AROUND THE REDUCTIONS

We denote by  $\mathcal{M}_a^{\text{ASD}}(E) \subset \mathcal{B}_a(E)$  the moduli of projectively ASD  $a$ -oriented connections in  $E$ , i.e. the moduli space

$$\mathcal{M}_a^{\text{ASD}}(E) := \mathcal{A}_a^{\text{ASD}}(E) / \mathcal{G}_E, \quad \mathcal{A}_a^{\text{ASD}}(E) := \{A \in \mathcal{A}_a(E) \mid (F_A^0)^+ = 0\}.$$

In the first subsection we will study the intersection of this moduli space with a normal neighborhood  $\mathcal{V}^\lambda$  of the reduction locus  $\mathcal{B}_a^\lambda(E) := \mathcal{A}_a^\lambda(E) / \mathcal{G}_E$ . We will see that, in a neighborhood of a reduction locus, the instanton moduli problem is equivalent to an *abelian* moduli problem, which is very much similar to the Seiberg-Witten moduli problem. We will denote by  $\mathcal{M}_a^\lambda(E)$  the subspace  $\mathcal{M}_a^\lambda(E) := \mathcal{M}_a^{\text{ASD}}(E) \cap \mathcal{B}_a^\lambda(E)$  of  $\lambda$ -reducible instantons.

**3.1. An abelian gauge theoretical problem.** Let  $L$  be a Hermitian line bundle which is isomorphic to a line subbundle of  $E$ , and put  $S := L^{\otimes 2} \otimes D^\vee$ . Consider the moduli space  $\mathcal{M}_a(L)$  of solutions of the system

$$(10) \quad \begin{cases} F_b^+ - \frac{1}{2}F_a^+ &= (\alpha \wedge \bar{\alpha})^+ \\ (d_{b^{\otimes 2} \otimes a^\vee}^*, d_{b^{\otimes 2} \otimes a^\vee}^+) \alpha &= 0 \end{cases}$$

for pairs  $(b, \alpha) \in \mathcal{A}(L) \times A^1(S)$ , modulo the abelian gauge group  $\mathcal{G} = \mathcal{C}^\infty(M, S^1)$ . This system is very much similar to the Seiberg-Witten system; indeed, the left hand operator in the first equation is elliptic and can be written as coupled Dirac operator. The main difference is that in general there is no a priori bound for the  $\alpha$ -component on the space of solutions of this system and, in general, the moduli space is not compact.

We denote by  $\mathcal{M}_a^{\text{red}}(L)$ ,  $\mathcal{M}_a^*(L)$  the subspaces of reducible (respectively irreducible) solutions. As in Seiberg-Witten theory “reducible pair” means “pair with trivial  $\alpha$ -component”. Therefore one has a natural identification

$$\mathcal{M}_a^{\text{red}}(L) \simeq T_a(L) := \mathcal{A}_a(L) / \mathcal{G}, \quad \text{where } \mathcal{A}_a(L) := \left\{ b \in \mathcal{A}(L) \mid F_b^+ = \frac{1}{2}F_a^+ \right\},$$

The space  $T_a(L)$  is either empty, or a  $b_1(M)$ -dimensional torus (when the harmonic representative of  $c_1(S) = 2c_1(L) - c_1(D)$  is ASD).

Put  $l := c_1(L)$  and  $\lambda = \{l, d - l\}$  and fix an isomorphism  $E = L \oplus (D \otimes L^\vee)$ . The map

$$\psi : (b, \alpha) \mapsto A_{b, \alpha} := \begin{pmatrix} d_b & \alpha \\ -\bar{\alpha} & d_{a \otimes b^\vee} \end{pmatrix} \in \mathcal{A}_a(E)$$

descends to a map  $\mathcal{M}_a(L) \rightarrow \mathcal{M}_a^{\text{ASD}}(E)$ . The image of this map is the subspace consisting of those instantons which can be brought in Coulomb gauge with respect to a  $\lambda$ -reducible connection. Suppose again that  $2l \neq d$  (such that any reduction having a parallel line subbundle of Chern class  $l$  is simple), and consider the continuous, gauge invariant function  $\varepsilon : \mathcal{A}_a^\lambda(E) \rightarrow \mathbb{R}_{>0}$  defining the normal neighborhood  $\mathcal{U}^\lambda$  of  $\mathcal{A}_a^\lambda(E)$  (see section 1.3). For  $b \in \mathcal{A}(L)$  put

$$\varepsilon(b) := \varepsilon(A_b) \text{ , where } A_b := b \oplus (a \otimes b^\vee) \in \mathcal{A}_a^\lambda(E) \text{ ,}$$

and denote  $\mathcal{W}^\lambda$  the  $\mathcal{G}$ -invariant subspace of the configuration space  $\mathcal{A}(L) \times A^1(S)$  defined by the inequality  $\|\alpha\|_{L^\infty} \leq \varepsilon(b)$ , and by  $\mathcal{Z}^\lambda$  its  $\mathcal{G}$ -quotient.

**Proposition 3.1.** *The restriction of the instanton moduli problem to the normal neighborhood  $\nu(\mathcal{U}^\lambda)$  of  $\mathcal{A}_a^\lambda(E)$  and the restriction of the abelian moduli problem (10) to the neighborhood  $\mathcal{W}^\lambda$  of  $\mathcal{A}(L) \times \{0\}$  are equivalent moduli problems.*

*In particular, the map induced by  $\psi$  applies  $\mathcal{M}_a(L) \cap \mathcal{Z}^\lambda$  isomorphically onto the neighborhood  $\mathcal{M}_a^{\text{ASD}}(E) \cap \mathcal{V}^\lambda$  of the subspace  $\mathcal{M}_a^\lambda(E)$  of  $\lambda$ -reducible instantons, and induces an isomorphism  $\mathcal{M}_a^{\text{red}}(L) \simeq \mathcal{M}_a^\lambda(E)$ .*

**Proof:** This follows directly from Proposition 2.1, (2). ■

**Corollary 3.2.** *Let  $b \in \mathcal{A}_a(L)$ . The deformation elliptic complex  $\mathcal{C}_{A_b}$  at the corresponding reduction  $A_b$  splits as a direct sum  $\mathcal{C}_{A_b} = \mathcal{C}_0^+(M) \oplus \mathcal{C}_{b^{\otimes 2} \otimes a^\vee}^+(S)$ , where  $\mathcal{C}_0^+(M)$  is the standard  $d^+$ -elliptic complex for  $i\mathbb{R}$ -valued forms and  $\mathcal{C}_{b^{\otimes 2} \otimes a^\vee}^+(S)$  is the  $d^+$ -elliptic complex for  $S$ -valued forms associated with the connection  $b^{\otimes 2} \otimes a^\vee$ .*

**Remark.** *The complex index of the elliptic complex  $\mathcal{C}_{b^{\otimes 2} \otimes a^\vee}^+(S)$  is given by*

$$\text{ind}_{\mathbb{C}}(C^+(S)) = c_1^2(S) + (b_+(M) - b_1(M) + 1) \text{ .}$$

**3.2. A twisted abelian gauge theoretical problem.** Let  $\rho : \pi_1(M, x_0) \rightarrow \mathbb{Z}_2$  be a group epimorphism and consider the associated objects  $\pi_\rho : M_\rho \rightarrow M$ ,  $\iota : M_\rho \rightarrow M_\rho$  introduced in section 1.4.

Let  $L$  be a Hermitian line bundle on  $M_\rho$  whose Chern class  $l$  is a solution of the system (7) and put  $\lambda = \{l, \iota^*(l)\}$ . In this section we will assume that  $l \neq \iota^*(l)$ . We fix an isomorphism  $\pi_\rho^*(E) = L \oplus \iota^*(L)$ . We denote by  $\mathcal{A}_\iota(L)$  the subspace of  $\mathcal{A}(L)$  consisting of connections  $b \in \mathcal{A}(L)$  such that  $b \otimes \iota^*(b) = \pi_\rho^*(a)$ . The natural gauge group acting on this space of connections is

$$\mathcal{G}_\iota := \{f \in \mathcal{C}^\infty(M_\rho, S^1) \mid \iota^*(f) = \bar{f}\} \text{ .}$$

The Lie algebra of this group can be identified with  $A^0(R_\rho)$ , where  $R_\rho$  is the real line bundle associated with the representation  $\rho$ . One has a natural embedding  $\mathcal{G}_\iota \rightarrow \mathcal{G}_{\pi_\rho^*(E)}$  factorizes through an embedding  $\mathcal{G}_\iota \hookrightarrow \mathcal{G}_E$ .

One has a  $\mathcal{G}_\iota$ -equivariant map  $\mathcal{A}_\iota(L) \rightarrow \mathcal{A}_a^\lambda(E)$  given by  $b \mapsto A_b$ , where  $A_b$  is the unique  $\rho$ -twisted connection of type  $\lambda$  whose pull-back to  $M_\rho$  is  $b \oplus \iota^*(b)$ .

Put  $S = L \otimes \iota^*(L)^\vee = L^{\otimes 2} \otimes \pi_\rho^*(D)$ . This Hermitian line bundle comes with a tautological isomorphism  $\iota^*(S) = S^\vee = \bar{S}$ . We introduce the spaces of  $\iota$ -twisted  $S$ -valued forms by

$$A_\iota^k(S) := \{\alpha \in A^k(S) \mid \iota^*(\alpha) = -\bar{\alpha}\} \text{ .}$$

Note that, for a connection  $b \in \mathcal{A}_\iota(L)$ , one has  $F_b - \frac{1}{2}\pi_\rho^*(F_a) \in A_\iota^2(S)$ . Endow  $M_\rho$  with the pull-back metric  $\pi_\rho^*(g)$ . Our abelian moduli problem is now

$$(11) \quad \begin{cases} F_b^+ - \frac{1}{2}\pi_\rho^*(F_a)^+ & = (\alpha \wedge \bar{\alpha})^+ \\ (d_{b \otimes 2 \otimes \pi_\rho^*(a)^\vee}^* + d_{b \otimes 2 \otimes \pi_\rho^*(a)^\vee}^+) \alpha & = 0 . \end{cases}$$

for pairs  $(b, \alpha) \in \mathcal{A}_\iota(L) \times A_\iota^1(S)$ , modulo the abelian gauge group  $\mathcal{G}_\iota$ .

The stabilizer of *any* pair  $(b, \alpha) \in \mathcal{A}_\iota(L) \times A_\iota^1(S)$  is  $\{\pm 1\} \subset \mathcal{G}_\iota$ , so, in fact, this twisted abelian moduli problem has no reductions at all. However, the moduli space  $\mathcal{M}_{a,\iota}(L)$  of solutions has two distinguished gauge invariant subspaces  $\mathcal{M}_{a,\iota}^{\text{red}}(L)$ ,  $\mathcal{M}_{a,\iota}^*(L)$  consisting of classes of solutions with vanishing (respectively non-vanishing)  $\alpha$ -component. As in the previous section we obtain a map  $\psi : \mathcal{M}_{a,\iota}(L) \rightarrow \mathcal{M}_a^{\text{ASD}}(E)$  which applies isomorphically  $\mathcal{M}_{a,\iota}^{\text{red}}(L)$  onto the space  $\mathcal{M}_a^\lambda(E)$  of  $\rho$ -twisted reducible instantons of type  $\lambda$ .

The space  $\mathcal{M}_{a,\iota}^{\text{red}}(L)$  has a simple geometric interpretation: one has an obvious identification

$$\mathcal{M}_{a,\iota}^{\text{red}}(L) \simeq \mathcal{A}_{\pi_\rho^*(a)}(L) \cap \mathcal{A}_\iota(L) / \mathcal{G}_\iota \subset \mathcal{M}_{\pi_\rho^*(a)}^{\text{red}}(L)$$

which shows that  $\mathcal{M}_{a,\iota}^{\text{red}}(L)$  is the subspace of  $\mathcal{M}_{\pi_\rho^*(a)}^{\text{red}}(L)$  defined by the equation  $[b \otimes \iota^*(b)] = [a]$ . If the harmonic representative of  $c_1(S)$  is not ASD (with respect to the pull-back metric), the space  $\mathcal{M}_{a,\iota}^{\text{red}}(L)$  will be empty. When this representative is ASD,  $\mathcal{M}_{\pi_\rho^*(a)}^{\text{red}}(L)$  will be a (non-empty!) subtorus of the torus  $\mathcal{M}_{\pi_\rho^*(a)}^{\text{red}}(L)$ ; this subtorus is (non-canonically) isomorphic to the quotient

$$iH^1(M_\rho, \mathbb{R})_\iota / 2\pi i H^1(M_\rho, \mathbb{Z})_\iota ,$$

where the symbol  $(-)_\iota$  means  $\iota$ -twisted, i.e. the subspace of  $(-)$  consisting of solutions of the equation  $\iota^*(x) = -x$ . In particular, if  $\iota^* : H^1(M_\rho, \mathbb{Z}) \rightarrow H^1(M_\rho, \mathbb{Z})$  is the identity and  $b_+(M_\rho) = 0$ , this space consists of a single point.

Consider the continuous, gauge invariant function  $\varepsilon : \mathcal{A}_a^\lambda(E) \rightarrow \mathbb{R}_{>0}$  defining the normal neighborhood  $\mathcal{U}^\lambda$  (see section 1.4) of  $\mathcal{A}_a^\lambda(E)$  and, for  $b \in \mathcal{A}(L)$  put  $\varepsilon(b) := \varepsilon(A_b)$ . As in the previous section we introduce the space  $\mathcal{W}^\lambda \subset \mathcal{A}_\iota(L) \times A_\iota^1(S)$  defined by the inequality  $\|\alpha\|_{L^\infty} < \varepsilon(b)$  and its gauge quotient  $\mathcal{Z}^\lambda = \mathcal{W}^\lambda / \mathcal{G}_\iota$ . We obtain:

**Proposition 3.3.** *The restriction of the instanton moduli problem to the normal neighborhood  $\nu(\mathcal{U}^\lambda)$  of the space of  $\rho$ -twisted, type  $\lambda$ -reductions  $\mathcal{A}_a^\lambda(E)$  and the restriction of the abelian moduli problem (11) to the neighborhood  $\mathcal{W}^\lambda$  of  $\mathcal{A}(L) \times \{0\}$  are equivalent moduli problems.*

*In particular, the map induced by  $\psi$  applies  $\mathcal{M}_{a,\iota}(L) \cap \mathcal{Z}^\lambda$  isomorphically onto the neighborhood  $\mathcal{M}_a^{\text{ASD}}(E) \cap \mathcal{V}^\lambda$  of the subspace  $\mathcal{M}_a^\lambda(E)$  of  $\rho$ -twisted,  $\lambda$ -reducible instantons, and induces an isomorphism  $\mathcal{M}_{a,\iota}^{\text{red}}(L) \simeq \mathcal{M}_a^\lambda(E)$ .*

**Corollary 3.4.** *Let  $b \in \mathcal{A}_{\pi_\rho^*(a)}(L) \cap \mathcal{A}_\iota(L)$ . The deformation complex  $\mathcal{C}_{A_b}$  at the corresponding  $\rho$ -twisted reduction  $A_b$  splits as a direct sum  $\mathcal{C}_{A_b} = \mathcal{C}_{0,\iota}^+(M_\rho) \oplus \mathcal{C}_{b \otimes \iota^*(b)^\vee, \iota}^+(S)$ , where  $\mathcal{C}_{0,\iota}^+(M_\rho)$  is the  $d^+$ -elliptic complex*

$$0 \longrightarrow iA_\iota^0(M_\rho) \longrightarrow iA_\iota^1(M_\rho) \xrightarrow{d^+} iA_{+,\iota}^2(M_\rho) \longrightarrow 0$$

*of imaginary  $\iota$ -twisted forms, and  $\mathcal{C}_{b \otimes \iota^*(b)^\vee, \iota}^+(S)$  is the  $d^+$ -elliptic complex of  $\iota$ -twisted  $S$ -valued forms associated with the connection  $b \otimes \iota^*(b)^\vee$ .*

The indices of the two elliptic complexes can be computed easily: The dimension  $h_\ell^k$  of the  $k$ -th harmonic space of  $\mathcal{C}_{0,\ell}^+(M_\rho)$  is

$$h_\ell^0 = 0, \quad h_\ell^1 = b_1(M_\rho) - b_1(M), \quad h_\ell^2 = b_+(M_\rho) - b_+(M),$$

Taking into account that  $\text{ind}(\mathcal{C}_0^+(M_\rho)) = 2\text{ind}(\mathcal{C}_0^+(M))$ , we get

$$\text{ind}(\mathcal{C}_{0,\ell}^+(M_\rho)) = b_+(M_\rho) - b_1(M_\rho) - (b_+(M) - b_1(M)) = b_+(M) - b_1(M) + 1.$$

For the complex  $\mathcal{C}_{b \otimes \iota^*(b)^\vee, \ell}^+(S)$ , note that the complex  $\mathcal{C}_{b \otimes \iota^*(b)^\vee}^+(S)$  splits as a direct sum

$$\mathcal{C}_{b \otimes \iota^*(b)^\vee}^+(S) = \mathcal{C}_{b \otimes \iota^*(b)^\vee, 0}^+(S) \oplus \mathcal{C}_{b \otimes \iota^*(b)^\vee, \ell}^+(S)$$

where  $\mathcal{C}_{b \otimes \iota^*(b)^\vee, 0}^+(S)$  is the  $d^+$ -complex of  $S$ -valued forms  $\alpha$  on  $M_\rho$  satisfying the relation  $\iota^*(\alpha) = \bar{\alpha}$ . Multiplication by  $i$  defines a real isomorphism  $\mathcal{C}_{b \otimes \iota^*(b)^\vee, 0}^+(S) \simeq \mathcal{C}_{b \otimes \iota^*(b)^\vee, \ell}^+(S)$ . Therefore

$$\text{ind}[\mathcal{C}_{b \otimes \iota^*(b)^\vee, \ell}^+(S)] = \text{ind}_{\mathbb{C}} \mathcal{C}^+(S) = c_1^2(S) + 2(b_+(M) - b_1(M) + 1).$$

**3.3. Generic regularity at the reductions.** The purpose of this section is to prove a strong generic regularity result for reducible instantons. We agree to call *regular* any solution  $A \in \mathcal{A}_a^{\text{ASD}}(E)$  (irreducible or reducible) with  $\mathbb{H}_A^2 = 0$ . Our result allows to prove that, under certain cohomological conditions on our data, there exists a *connected*, dense, open set of metrics for which no irregular reduction appears in the moduli space. This will allow us in the next section to introduce Donaldson type invariants for definite manifolds, even in the cases when non-empty reduction loci are present in the moduli space.

Let  $M$  be a 4-manifold with  $b_+(M) = 0$ , and  $E$  a Hermitian bundle of rank 2 on  $M$ ; put as usual  $D := \det(E)$ ,  $d := c_1(D)$ ,  $c = c_2(E)$ . Let  $\lambda = \{l, d - l\}$  be a topological decomposition of  $E$  with  $2l \neq d$ . The second cohomology of the deformation elliptic complex of a  $\lambda$ -reducible instanton reduces to the second cohomology of the elliptic complex  $\mathcal{C}_{b \otimes 2 \otimes a^\vee}^+(S)$ , where  $S = L^{\otimes 2} \otimes D^\vee$ .

Denote by  $\text{Met}$  ( $\text{Met}^r$ ) the space of smooth (respectively class  $\mathcal{C}^r$ ) Riemannian metrics on  $M$ , where  $r \gg k$ . Our first result is a transversality theorem (with respect to variations of the metric  $g$ ) for the complement  $\mathcal{H}_g(S)^*$  of the zero section in the complex linear space

$$\mathcal{H}_g(S) := \coprod_{[\sigma] \in \mathcal{M}^{\text{ASD}}_g(S)} [\mathbb{H}^0(\mathcal{C}_\sigma^{+g}(S)) \oplus \mathbb{H}^2(\mathcal{C}_\sigma^{+g}(S))]$$

over the torus  $\mathcal{M}^{\text{ASD}}_g(S)$ . Unfortunately this transversality result can be proved only for metrics having the following property

**H(S)** : *The vanishing locus of the  $g_0$  harmonic representative of  $c_1^{\text{DR}}(S)$  has Hausdorff dimension  $\leq 2$ .*

This condition is satisfied by any  $\mathcal{C}^\infty$ -metric by a result of Bär [B], and any  $c_1(S)$ -admissible metric  $g \in \text{Met}_{\geq 2}^r(c_1^{\text{DR}}(S))$  (see section 5.2 in the Appendix).

The space  $\mathcal{H}_g(S)^*$  can be identified with the  $\mathcal{G}_{x_0}$ -quotient of the space of solutions of the system

$$(12) \quad \begin{cases} F_\sigma^{+g} & = 0 \\ d_\sigma \zeta + d_{\sigma^*g}^* \eta & = 0 \end{cases}$$

for triples  $(\sigma, \zeta, \eta) \in \mathcal{A}(S) \times [(A^0(S) \oplus A_{+g}^2(S)) \setminus \{0\}]$ .

Regard (12) as an equation for systems  $(g, \sigma, \zeta, \eta)$ , where  $g \in \text{Met}^r$ ,  $\sigma \in \mathcal{A}(S)_k$ , and  $(\zeta, \eta) \in [(A^0(S)_k \oplus A_{+g}^2(S)_k) \setminus \{0\}]$ . Therefore our configuration space is now

$$\mathcal{A}_k^* := \mathcal{A}(S)_k \times [\mathcal{A}^0(S)_k \oplus \mathcal{A}_+^2(S)_k]^* ,$$

where  $\mathcal{A}^0(S)_k$  is the trivial bundle  $\text{Met}^r \times \mathcal{A}^0(S)_k$  over  $\text{Met}^r$ ,  $\mathcal{A}_+^2(S)_k$  is the bundle of  $S$ -valued selfdual forms over  $\text{Met}^r$  (see section 5.1 in the Appendix), and the symbol  $[\cdot]^*$  on the right stands for the complement of the zero-section. Regarding  $\mathcal{A}_k^*$  as a locally trivial Banach bundle over the Banach manifold  $\text{Met}^r$ , and denoting by  $p : \mathcal{A}_k^* \rightarrow \text{Met}^r$  the obvious projection, we see that the left hand terms of the equations (12) define sections  $u, v$  in the bundles  $p^*(i[\mathcal{A}_+^2]_{k-1})$  and  $\mathcal{A}_k^* \times A^1(S)_{k-1}$  over  $\mathcal{A}_k^*$ .

**Theorem 3.5.** *The section  $(u, v)$  in the bundle  $[p^*(i[\mathcal{A}_+^2]_{k-1})] \oplus A^1(S)_{k-1}$  over  $\mathcal{A}_k$  is submersive at any solution  $(g_0, \sigma_0, \zeta_0, \eta_0)$  where  $g_0$  has the property  $\mathbf{H}(S)$  and  $(\zeta_0, \eta_0) \neq 0$ .*

**Proof:** Note first that, under our assumptions, the connection  $\sigma$  cannot admit nontrivial parallel sections, so applying  $d_{\sigma_0}^*$  to the second equation, we get  $\zeta_0 = 0$ . The hypothesis gives  $\eta_0 \neq 0$ . Use the metric  $g_0$  as a background metric to parameterize the manifold  $\text{Met}^r$  and to trivialize the bundles  $i[\mathcal{A}_+^2]_{k-1}$ ,  $\mathcal{A}_+^2(S)_k$  over this manifold (see section 5.1). The system (12) is equivalent with

$$(13) \quad \begin{cases} [(h^{-1})^*(F_\sigma)]^+ & = 0 \\ d_\sigma \zeta - *_{g_h} d_\sigma [h^*(\eta)] & = 0 , \end{cases}$$

where now the upper script  $+$  is used for the selfdual projection with respect to  $g_0$ , and  $\eta \in A_+^2(S)_k$ . The left hand terms of the equations define a smooth map

$$\Gamma(\text{Sym}^+(T_M, g_0))^r \times \mathcal{A}(S)_k \times [(A^0(S)_k \times A_+^2(S)_k) \setminus \{0\}] \xrightarrow{(U, V)} i[\mathcal{A}_+^2]_{k-1} \times A^1(S)_{k-1} .$$

Our task is to prove that the differential of this map at  $(\text{id}, \sigma_0, 0, \eta_0)$  is surjective. Let  $(\alpha, \beta) \in i[\mathcal{A}_+^2]_{k-1} \times A^1(S)_{k-1}$  be a pair which is  $L^2$ -orthogonal to the range of this differential. Using variations of the variables  $\zeta$  and  $\eta$  (for  $h = \text{id}$ ) we get

$$(14) \quad d_{\sigma_0}^* \beta = 0 , \quad d_{\sigma_0}^+ \beta = 0 .$$

Using the notations of section 5.1 in Appendix, one has

$$\frac{\partial U}{\partial h} \Big|_{(\text{id}, \sigma_0, 0, \eta_0)} (\chi) = -m_-^+(\chi)(F_{\sigma_0}) , \quad \frac{\partial V}{\partial h} \Big|_{(\text{id}, \sigma_0, 0, \eta_0)} (\chi) = - * d_{\sigma_0}(m(\chi)\eta_0)$$

for any  $g_0$ -symmetric endomorphism  $\chi$ . For the second formula we used the fact that  $d_{\sigma_0}\eta_0 = 0$  (recall that  $\zeta_0 = 0$  and  $*\eta_0 = \eta_0$ ), so the term containing the differential of  $*_{g_h}$  with respect to  $h$  vanishes. Therefore, using variations  $\chi \in A^0(\text{Sym}(T_M, g_0))^r$  of  $h$  with the property

$$m_-^+(\chi)(F_{\sigma_0}) = 0 ,$$

and noting that  $m_+^-(\chi) = [m_-^+(\chi)]^*$  (see section 5.1), we obtain for any such  $\chi$

$$(15) \quad 0 = \langle *d_{\sigma_0}(m(\chi)\eta_0), \beta \rangle_{L^2} = -\langle m(\chi)\eta_0, *d_{\sigma_0}\beta \rangle_{L^2} =$$



$$= \langle m_+^-(\chi)\eta_0, d_{\sigma_0}\beta \rangle_{L^2} = \langle [m_+^-(\chi)]^*\eta_0, d_{\sigma_0}\beta \rangle_{L^2} .$$

Here we used the fact that the 2-form  $d_{\sigma_0}\beta$  is ASD. In all these formulae  $\langle \cdot, \cdot \rangle$  stands for the real (Euclidean) inner product. By Lemma 5.1 in Appendix we see that *any* homomorphism  $m \in A^0(\text{Hom}(\Lambda_-^2, \Lambda_+^2))^r$  can be written as  $m_+^-(\chi)$  for a certain symmetric endomorphism  $\chi$ . Therefore (15) holds for any such section  $m$  for which  $m(F_{\sigma_0}) = 0$ . Now regard  $\beta$  as an element in  $A^0(\text{Hom}_{\mathbb{R}}(S_{\mathbb{R}}^{\vee}, \Lambda_-^2))_{k-1}$ ,  $\eta_0$  as an element in  $A^0(\text{Hom}_{\mathbb{R}}(S_{\mathbb{R}}^{\vee}, \Lambda_+^2))_k$ , and denote by  $[\eta_0]_{\mathbb{R}}^* \in A^0(\text{Hom}_{\mathbb{R}}(\Lambda_+^2, S_{\mathbb{R}}^{\vee}))_k$  its adjoint with respect to the obvious real inner products. Changing the position of  $\eta_0$  in (15) we obtain

$$\langle m^*, d_{\sigma_0}\beta \circ [\eta_0]_{\mathbb{R}}^* \rangle_{L^2} = 0 ,$$

for every  $m \in A^0(\text{Hom}(\Lambda_-^2, \Lambda_+^2))^r$  for which  $m(F_{\sigma_0}) = 0$ . The condition  $m(F_{\sigma_0}) = 0$  is equivalent to the condition  $\text{im}(m^*) \subset F_{\sigma_0}^{\perp}$ . Let  $U$  be the complement of the vanishing locus of  $F_{\sigma_0}$ . We conclude that  $d_{\sigma_0}\beta \circ [\eta_0]_{\mathbb{R}}^*|_U$  is  $L^2$ -orthogonal on the whole space of compactly supported  $F_{\sigma_0}^{\perp}$ -valued bundle homomorphisms  $\Lambda_+^2|_U \rightarrow \Lambda_-^2|_U$ . Therefore  $d_{\sigma_0}\beta \circ [\eta_0]_{\mathbb{R}}^*|_U$  takes values in the real line bundle generated by  $F_{\sigma_0}$ . This implies that either there exists a non-empty open subset  $V \subset U$  on which  $\eta_0$  has (real) rank at most 1, or  $d_{\sigma_0}\beta|_U$  takes values in real line bundle generated by  $F_{\sigma_0}|_U$ . In the first case we obtain  $\eta_0 = 0$  by Proposition 5.10 in the Appendix. This contradicts the assumption. In the second case one gets  $\beta = 0$  by Corollary 5.9 (which applies because  $g_0$  has the property  $\mathbf{H}(S)$ ). Finally, using variations of  $\sigma$  and the assumption  $b_+(M) = 0$ , we obtain  $\alpha = 0$ .  $\blacksquare$

Denote by  $\text{Met}_{\text{bad}}^r(\lambda)$  the subspace of metrics for which there exists a non-regular  $\lambda$ -reducible instanton. Let  $U \subset \text{Met}^r$  be any open subset of metrics satisfying the property  $\mathbf{H}(S)$ .

**Theorem 3.6.** *Suppose that  $(2l - d)^2 < 0$ . Then  $\text{Met}_{\text{bad}}^r(\lambda) \cap U$  is closed and nowhere dense in  $U$  and the natural morphism*

$$\pi_i(U \setminus \text{Met}_{\text{bad}}^r(\lambda)) \rightarrow \pi_i(U)$$

*is bijective for any  $i$  in the range  $0 \leq i \leq -2(2l - d)^2 + b_1(M) - 2$  and surjective for  $i = -2(2l - d)^2 + b_1(M) - 1$ .*

**Proof:** By Theorem 3.5 it follows that the section  $(u, v)$  is transversal at any solution with metric component in  $U$ , so the vanishing locus  $Z(u, v) \cap p^{-1}(U)$  is a smooth Banach manifold over  $U$ . The gauge quotient

$$\mathcal{H}(S)^*|_U := Z(u, v) \cap p^{-1}(U) / \mathcal{G}_{k+1, x_0}$$

will also be a smooth manifold, and the natural projection  $\mathcal{H}(S)^*|_U \rightarrow \text{Met}^r$  is Fredholm of real index  $2[c_1(S)^2 + (1 - b_1(M))] + b_1(M) = 2c_1(S)^2 - b_1(M) + 2$ . One has a natural  $\mathbb{C}^*$ -action on  $\mathcal{H}(S)^*$ , and the projection

$$\mathcal{H}(S)^* / \mathbb{C}^* \longrightarrow \text{Met}^r$$

will be Fredholm of real index  $2c_1(S)^2 - b_1(M)$ . It suffices to apply Lemma 5.7 in the Appendix.  $\blacksquare$

The same arguments can be used to prove regularity at a locus of twisted reductions (see sections 1.4, 3.2). However, there is an important detail here which

should be taken into account carefully: in general, for an epimorphism  $\pi(M) \rightarrow \mathbb{Z}_2$ , the condition  $b_+(M) = 0$  does *not* imply  $b_+(M_\rho) = 0$ . When  $b_+(M_\rho) = 0$ , one has  $h_l^2 = 0$ ,  $h_l^1 = b_1(M) - 1$  (see section 3.2), and one can obtain generic regularity at the  $\rho$ -twisted reduction in the same way as for non-twisted reductions, by extending our proofs to the twisted case. The main difference is that the “normal” elliptic complex”  $\mathcal{C}_{b \otimes \iota^*(b)^\vee, \iota}^+$  has no complex structure.

In the twisted case, one has a Hermitian line bundle  $S$  on  $M_\rho$  which comes with an isomorphism  $\iota^*(S) \simeq \bar{S}$ ; and for any metric  $g$  on  $M$  the  $\pi_\rho(g)$ -harmonic representative of  $c_1(S) = l - \iota^*(l)$  is  $\rho$ -equivariant, so the condition  $\mathbf{H}(S)$  has sense for  $g$ . Put  $\lambda := \{l, \iota^*(l)\}$  and denote by  $\text{Met}_{\text{bad}}^r(\lambda)$  the space of  $\mathcal{C}^r$ -metrics for which there exists a non-regular  $\rho$ -twisted reducible instanton of type  $\lambda$ . The result for the twisted case is

**Proposition 3.7.** *Suppose that  $b_+(M) = b_+(M_\rho) = 0$  and let  $l \in H^2(M_\rho, \mathbb{Z})$  a solution of the system (7) with  $(l - \iota^*(l))^2 < 0$ . Put  $\lambda := \{l, \iota^*(l)\}$  and let  $U \subset \text{Met}^r$  an open set of metrics having the property  $\mathbf{H}(S)$ . Then  $\text{Met}_{\text{bad}}^r(\lambda) \cap U$  is closed and the natural morphism*

$$\pi_i(U \setminus \text{Met}_{\text{bad}}^r(\lambda)) \rightarrow \pi_i(U)$$

is bijective for any  $i$  in the range  $0 \leq i \leq -(l - \iota^*(l))^2 + b_1(M) - 2$ , and surjective for  $i = -(l - \iota^*(l))^2 + b_1(M) - 1$ .

**Remark.** Note that  $-(l - \iota^*(l))^2 = 2[4c_2(E) - c_1(E)^2]$ , hence under our assumptions one has  $-(l - \iota^*(l))^2 \geq 2$ .

**Corollary 3.8.** *Let  $M$  be a 4-manifold, and  $E$  a Hermitian rank 2 bundle on  $M$ .*

(1) *Suppose that*

$$(16) \quad b_+(M) = 0 \text{ and } c_1(E) \notin 2H^2(M, \mathbb{Z}) + \text{Tors} .$$

*There exists a connected, dense, open subset  $\text{Met}_{\text{good}}^r(E) \subset \text{Met}^r$  such that, for any  $g \in \text{Met}_{\text{good}}^r(E)$ , the reductions in the Uhlenbeck compactification of the moduli space  $\mathcal{M}_{a,g}^{\text{ASD}}(E)$  of  $g$ -instantons are all regular.*

(2) *Suppose that (16) holds and for every epimorphism  $\rho : \pi_1(M, x_0) \rightarrow \mathbb{Z}_2$*

$$(17) \quad b_+(M_\rho) = 0 \text{ and } \pi_\rho^*(c_1(E)) \notin 2H^2(M_\rho, \mathbb{Z}) + \text{Tors} .$$

*There exists a connected, dense, open subset  $\text{Met}_{\text{vgood}}^r(E) \subset \text{Met}^r$  such that, for any  $g \in \text{Met}_{\text{vgood}}^r(E)$ , the reductions and the twisted reductions in the Uhlenbeck compactification of the moduli space  $\mathcal{M}_{a,g}^{\text{ASD}}(E)$  of  $g$ -instantons are all regular.*

**Proof:** 1. Denote by  $\Lambda(c)$  the finite set of unordered pairs  $\lambda = \{l, d-l\}$  satisfying  $l \cdot (d-l) \leq c$ . For  $c = c_2(E)$ , this set is the set of all topological decompositions of all bundles  $E'$  which must be considered in the construction of the Uhlenbeck compactification of  $\mathcal{M}_a^{\text{ASD}}(E)$ . Put

$$\text{Met}_{\text{good}}^r(E) := \text{Met}_{\text{adm}}^r(c) \setminus \left[ \bigcup_{\lambda \in \Lambda(c)} \text{Met}_{\text{bad}}^r(\lambda) \right]$$

(see section 5.2 in the Appendix).

2. One defines the space  $\text{Met}_{\text{vgood}}^r(E)$  in a similar way by replacing  $\text{Met}_{\text{adm}}^r(c)$  with  $\text{Met}_{\text{tadm}}^r(c)$  (see section 5.2), and removing the bad loci associated with all twisted

and non-twisted reductions of bundles  $E'$  with  $\det(E') = \det(E)$ ,  $c_2(E') \leq c_2(E)$ . ■

**Remark.** Suppose that  $b_+(M) = 0$ ,  $b_2(M) > 0$ , and  $b_+(M_\rho) = 0$  for any epimorphism  $\rho : \pi_1(M, x_0) \rightarrow \mathbb{Z}_2$ . Let  $d$  be an integral lift of  $w_2(M)$ . Then  $d \notin 2H^2(M, \mathbb{Z}) + \text{Tors}$  and  $\pi_\rho^*(d) \notin 2H^2(M_\rho, \mathbb{Z}) + \text{Tors}(H^2(M_\rho, \mathbb{Z}))$  for every epimorphism  $\rho$ , so Corollary 3.8 (2) applies for any bundle  $E$  with  $c_1(E) = d$ .

**Proof:** By Donaldson first theorem, the intersection form on  $H^2(M, \mathbb{Z})/\text{Tors}$  is trivial over  $\mathbb{Z}$ . Choosing an orthonormal basis  $(e_i)_{1 \leq i \leq b_2(M)}$  in this lattice, one obtains  $d \cdot e_i \equiv e_i^2 \equiv -1 \pmod{2}$ , so  $d$  cannot be divisible by 2 in  $H^2(M, \mathbb{Z})/\text{Tors}$ .

On the other hand, the class  $\pi_\rho^*(d)$  is an integral lift of  $\pi_\rho^*(w_2(M)) = w_2(M_\rho)$ . Since  $b_+(M_\rho) = b_+(M) = 0$ , one gets easily (comparing the signatures and the Euler characteristics of the two manifolds) that  $b_1(M_\rho) = 2b_1(M) - 1$  and  $b_2(M_\rho) = 2b_2(M) > 0$ . Therefore the same argument applies for  $M_\rho$ , proving that  $\pi_\rho^*(d)$  cannot be divisible by 2 in  $H^2(M_\rho, \mathbb{Z})/\text{Tors}$ . ■

**Remark.** Similar generic regularity results can be obtained using abstract perturbations of the ASD-equations around the reduction loci (see [DK] p. 156). However, since in our general framework the reductions are not necessarily isolated points in the moduli space, this method is more complicated than in the classical case. Moreover, for our purposes (see section 4) one must check that the perturbed moduli space still has a natural compactification, and that the ‘‘cobordism type’’ of this compactification is well defined.

Combing Corollary 3.8 with the classical transversality theorem for irreducible instantons [FU], [DK], one shows that

**Lemma 3.9.** *The subset  $\text{Met}_{\text{vgood}}^r(E) \subset \text{Met}_{\text{vgood}}^r(E)$  of metrics for which the moduli space  $\mathcal{M}_{a,g}^{\text{ASD}}(E)$  contains only regular solutions is dense of the second Baire category. This set is also open (but in general non-connected!) when  $\Delta(E) := 4c_2(E) - c_1(E)^2 \leq 3$ .*

**Proof:** The condition  $\Delta(E) \leq 3$  implies that the projection  $\tilde{\mathcal{M}}_a^{\text{ASD}}(E) \rightarrow \text{Met}^r$  of the parameterized instanton moduli space on the space of metrics is proper. The openness of  $\text{Met}_{\text{vgood}}^r(E)$  in  $\text{Met}_{\text{vgood}}^r(E)$  follows by elliptic semicontinuity. ■

## 4. APPLICATIONS

**4.1. Geometric properties of instanton moduli spaces on manifolds with  $b_+ = 0$ .** We will see that using our regularity results combined with the topological results obtained in sections 2.1, 2.2 one can obtain important information about the geometry of the ASD moduli spaces.

The purpose of this section is not to give an exhaustive list of all possible applications of this type, but only to illustrate the method with an explicit example, which came to my attention when I began to work on the classification of class VII surfaces with  $b_2 = 2$  [Te3].

Let  $M$  be a 4-manifold with the topological properties considered in the examples studied in section 2.2:  $H_1(M, \mathbb{Z}) \simeq \mathbb{Z}$ ,  $b_2(M) = b_-(M) = 2$ . Consider again a

rank 2-Hermitian bundle  $E$  on  $M$  with  $c_1(E) = d = e_1 + e_2$  (where  $(e_1, e_2)$  is an orthonormal basis of  $H^2(M, \mathbb{Z}) \simeq \mathbb{Z}^{\oplus 2}$ ) and  $c_2(E) = 0$ . As in section 2.2 put  $\lambda_0 := \{0, d\}$ ,  $\lambda_1 := \{e_1, e_2\}$  and note that  $\Lambda_E = \{\lambda_0, \lambda_1\}$ . Let  $L_0, L_1$  be Hermitian line bundles with Chern classes  $c_1(L_0) = 0$ ,  $c_1(L_1) = e_1$ , and put  $S_i := L_i^{\otimes 2} \otimes D^\vee$ .

The expected dimension of the instanton moduli space  $\mathcal{M}_a^{\text{ASD}}(E)$  is 4 and, since  $\Delta(E) := 4c_2(E) - c_1(E)^2 < 4$ , this moduli space is compact.  $\mathcal{M}_a^{\text{ASD}}(E)$  contains two circles of reductions  $\mathcal{M}_a^{\lambda_0}(E)$  and  $\mathcal{M}_a^{\lambda_1}(E)$ . An interesting application of our results is the following:

**Theorem 4.1.** *For every Riemannian metric  $g$  on  $M$  and abelian connection  $a \in \mathcal{A}(\det(E))$ , the two circles of reductions  $\mathcal{M}_a^{\lambda_0}(E)$ ,  $\mathcal{M}_a^{\lambda_1}(E)$  belong to the same connected component of the moduli space  $\mathcal{M}_a^{\text{ASD}}(E)$ .*

**Proof:** For a metric  $g \in \text{Met}_{w_{\text{good}}}^r(E)$  the moduli space  $\mathcal{M}_a^{\text{ASD}}(E)$  contains only regular solutions. Regularity at the reductions implies that the linear spaces

$$\mathcal{H}_i^1 := \bigcup_{[b] \in T_a(L)} \mathbb{H}^1(\mathcal{C}_{b^{\otimes 2} \otimes a^\vee}^+(S_i)) \rightarrow T_a(L) \simeq \mathcal{M}_a^{\lambda_i}(E)$$

are rank 2 complex vector bundle (see section 3.1).  $\mathcal{M}_a^{\lambda_i}(E)$  has a neighborhood  $\nu_i$  which can be identified with the  $S^1$ -quotient of an  $S^1$ -invariant neighborhood of the zero section of  $\mathcal{H}_i^1$ . Let  $\sigma$  be a generator of  $H_1(M, \mathbb{Z})$ . By Corollary 2.6 we see that the restriction of the Donaldson class  $\mu(\sigma)$  to the boundary  $\partial(\nu_i)$  coincides (up to sign) with the fundamental class of this 3-manifold. More precisely, if we choose an orientation of the moduli space as in [DK] p. 283, and we endow  $\partial(\nu_i)$  with the boundary orientations, we see that  $\mu(\sigma)|_{\partial(\nu_0)} = \pm[\partial(\nu_0)]$  and  $\mu(\sigma)|_{\partial(\nu_1)} = \mp[\partial(\nu_1)]$ . Therefore  $\partial(\nu_i)$  cannot be homologically trivial in  $\mathcal{B}_a^*(E)$ . This shows that the two boundaries (hence also the corresponding circles) belong to the same connected component.

To complete the proof for an arbitrary metric  $g$ , use the density of the space  $\text{Met}_{w_{\text{good}}}^r(E)$  and note that if the reduction circles  $\mathcal{M}_a^{\lambda_i}(E)$  belonged to different connected components, the same would happen for any metric  $g'$  sufficiently close to  $g$ .  $\blacksquare$

**Example:** Consider the 4-manifold  $M = (S^1 \times S^3) \# \bar{\mathbb{P}}^2 \# \bar{\mathbb{P}}^2$ . This manifold has the differentiable type of a Hopf surface blown up at two points. It is convenient to endow  $M$  with the complex structure of a *minimal* class VII surface with  $b_2 = 2$ . Choosing the Gauduchon metric in a convenient way and using the Kobayashi-Hitchin correspondence to identify instantons with polystable bundles, one obtains (see [Te3]):

$$\mathcal{M}_a^{\text{ASD}}(E) \simeq S^4,$$

so (despite the presence of the reductions) the moduli space gets an obvious smooth structure on the moduli space. The two reduction circles  $\mathcal{M}_a^{\lambda_i}(E)$  are smoothly embedded in the sphere.

**4.2. New Donaldson invariants.** In this section we introduce a new class of Donaldson type invariants, which are defined for definite 4-manifolds. Note that Donaldson type invariants for  $\mathbb{Z}[\mathbb{Z}]$  homology  $S^1 \times S^3$ -manifolds have already been considered in [RS1], [FuO]. Moreover, Rubermann and Saveliev introduced recently

a new differential topological invariant – based on the Dirac equation – for a special class of 4-manifolds with  $b_2 = 0$  and  $b_1 = 1$  [RS2].

4.2.1. *Low energy Donaldson invariants. Casson type invariants.* Let  $M$  be a negative definite 4-manifold, and let  $(e_1, \dots, e_{b_2(M)})$  be an orthonormal basis in  $H^2(M, \mathbb{Z})/\text{Tors}$ . Let  $d \in H^2(M, \mathbb{Z})$  be a lift of  $e_1 + \dots + e_{b_2(M)}$  and denote by  $\bar{d}$  its image in  $H^2(X, \mathbb{Z}_2)$ . Note that one must have  $\bar{d} = w_2(M)$ , when  $H^2(M, \mathbb{Z})$  is torsion free.

Let  $E$  be a Hermitian 2-bundle on  $M$  with  $c_1(E) = d$  and put as usual  $D := \det(E)$ . If  $\{l, d - l\}$  is a topological decomposition of  $E$ , then, writing  $l = \sum l_i e_i$ , with  $l_i \in \mathbb{Z}$ , one gets

$$c_2(E) = \sum l_i(l_i - 1) \geq 0 .$$

Therefore, for  $c_2(E) < 0$ , the bundle  $E$  admits no topological decomposition. On the other hand the expected dimension of the Donaldson moduli space  $\mathcal{M}_a^{\text{ASD}}(E)$  is

$$\delta = 2(4c_2(E) + b_2(M)) + 3(b_1(M) - 1) .$$

If  $c_2(E)$  is chosen such that  $\Delta(E) = 4c_2(E) + b_2(M) \in \{0, 1, 2, 3\}$  the corresponding moduli space will be a priori compact (i.e. compact independently of the metric). When  $b_2(M) \geq 4$ , the corresponding values of  $c_2(E)$  are negative. When  $b_1(M) \geq 1$  the corresponding expected dimension will be non-negative. Therefore:

**Remark.** *Suppose that  $b_1(M) \geq 1$  and  $b_2(M) \geq 4$ , and choose  $c_2(E) := -\left\lfloor \frac{b_2(M)}{4} \right\rfloor$ . The corresponding moduli space will be a priori compact, of non-negative expected dimension  $2\Delta(E) + 3(b_1 - 1)$  and will contain no reduction.*

In other words, for this special value of  $c_2(E)$ , one can define very easily Donaldson type invariants by evaluating products of classes of the form  $\mu(h)$  on the virtual fundamental class of the moduli space [Br]. In this case, one does not really need regular moduli spaces, because the formalism of virtual fundamental classes gives directly a well defined homology class in the space  $\mathcal{B}_a^*(E)$ .

A very interesting case is when  $b_2(M) \geq 4$  is divisible by 4. In this case this special value of  $c_2(E)$  is  $-\frac{b_2(M)}{4}$  and the corresponding discriminant  $\Delta(E)$  vanishes. Therefore, in this case  $\mathcal{M}^{\text{ASD}}(E)$  coincides with the *moduli space of  $PU(2)$ -representations of  $\pi_1(M, x_0)$  with fixed Stiefel-Whitney class  $\bar{d}$ , modulo  $SU(2)$ -conjugation*. The invariants associated with such a moduli space should be called four-dimensional Casson type invariants. They should be regarded as an extension of the similar  $SU(2)$ -invariant defined for  $\mathbb{Z}[\mathbb{Z}]$ -homology  $S^1 \times S^3$ -manifolds (see [RS1], [FuO]) to our new class of homology types. Note that, because of the absence of reductions in our moduli space, the definition of the invariant in our case is much easier.

Note the following simple vanishing result, which shows that, if non-trivial, this Casson type invariants can be regarded as obstructions to the representability of the basis elements  $e_i$  by 2-spheres.

**Remark.** *Suppose that  $b_2(M)$  is divisible by 4 and one of the basis elements  $e_i$  is represented by a continuous map  $S^2 \rightarrow M$ . Then the moduli space associated with the Chern class  $c_2(E) = -\frac{b_2(M)}{4}$  is empty. In particular, the corresponding Casson type invariants vanish.*

**Proof:** If  $f : S \rightarrow M$  represents  $e_i$  (for an oriented closed surface  $S$ ), one has

$$\langle f^*(\bar{d}), [S] \rangle = \langle \bar{d}, f_*([S]) \rangle = \langle d, f_*([S]) \rangle \pmod{2} = \langle d, e_i \rangle \pmod{2} = 1 .$$

Let  $x_0 \in S$ , and  $\rho : \pi_1(M, f(x_0)) \rightarrow PU(2)$  a representation of Stiefel-Whitney class  $\bar{d}$ . The composition  $\pi_1(S, x_0) \rightarrow \pi_1(M, f(x_0)) \rightarrow PU(2)$  will be a representation with Stiefel-Whitney class  $f^*(\bar{d}) \neq 0$ , so  $S$  cannot be simply connected. ■

Interestingly, one has:

**Remark.** *There exist definite negative 4-manifolds, with the property that no element  $e \in H^2(M, \mathbb{Z})/\text{Tors}$  with  $e^2 = -1$  can be represented by a continuous map  $f : S^2 \rightarrow M$ .*

Indeed, it suffices to consider a fake projective plane (see for instance [PY]) with reversed orientation. Since the universal cover of such a 4-manifold is the complex 2-ball, we see that the generator of its homology cannot be represented by a continuous map  $f : S^2 \rightarrow M$ .

Consider now the case

$$c_2(E) \in \left( - \left[ \frac{b_2(M)}{4} \right], -1 \right] .$$

In this range, one loses “a priori compactness”, but has moduli spaces with no reductions in their Uhlenbeck compactifications. In this range, one uses Donaldson’s method [DK] to define the invariants geometrically: one uses metrics for which all strata are regular, constructs distinguished cycles representing the  $\mu$ -classes (and which extends to the Uhlenbeck compactification) and defines the invariants by intersecting the moduli space with systems of such cycles (which can be chosen so that they intersect transversally in the main stratum).

4.2.2. *Invariants associated with classes in  $H^*(\mathcal{B}^*, \mathcal{V}^*)$ . Invariants defined using the cobordism type of the moduli space.* We illustrate these types of invariants in the concrete situation considered in sections 2.2 and 4.1: a negative definite 4-manifold with  $H_1(M, \mathbb{Z}) \simeq \mathbb{Z}$ ,  $b_2(M) = b_-(M) = 2$  endowed with a rank 2-Hermitian bundle  $E$  on  $M$  with  $c_1(E) = d = e_1 + e_2$  (where  $(e_1, e_2)$  is an orthonormal basis of  $H^2(M, \mathbb{Z}) \simeq \mathbb{Z}^{\oplus 2}$ ) and  $c_2(E) = 0$ .

The moduli space is a priori compact, but it always contains two circles of reductions. We have two ways to define invariants in this situation:

1. Use a generic metric in the sense of [DK], for which the irreducible part of the moduli space is regular. Regard the (oriented) moduli space  $\mathcal{M}_a^{\text{ASD}}(E)^*$  of irreducible instantons as a cycle in the relative homology  $H_4(\mathcal{B}^*, \mathcal{V}^*)$ . On the other hand, we have seen in section 2.2 that the relative rational cohomology  $H^4(\mathcal{B}^*, \mathcal{V}^*) \simeq \mathbb{Q}^4$  and that this group fits in a short exact sequence

$$(18) \quad 0 \longrightarrow \frac{H^3(\mathcal{V}_0^*) \oplus H^3(\mathcal{V}_1^*)}{\langle (\delta_{L_0}(\sigma_0)h_{u_0}, \delta_{L_1}(\sigma_0)h_{u_1}) \rangle} \longrightarrow H^4(\mathcal{B}^*, \mathcal{V}^*) \longrightarrow \langle \mu(*) + \mu(f_1)^2, \mu(*) + \mu(f_2)^2, \mu(\lambda_0)\mu(\sigma_0) \rangle_{\mathbb{Q}} \rightarrow 0 .$$

Evaluating classes in  $H^4(\mathcal{B}^*, \mathcal{V}^*)$  on the relative homology class defined by the moduli space  $\mathcal{M}_a^{\text{ASD}}(E)^*$  one gets well-defined invariants. Note however that, since

the exact sequence (18) does not split canonically, one cannot parameterize this set of invariants in an obvious way.

2. For a metric  $g \in \text{Met}_{\text{wgood}}^r(E)$  all solutions (including the reductions) in the moduli space are regular (see Lemma 3.9). The main observation here is that each reduction circle  $\mathcal{M}_a^{\lambda_i}(E)$  has a neighborhood isomorphic to the  $S^1$ -quotients of a neighborhood of the zero section in a rank 2 complex bundle  $\mathcal{H}_i^1$  over  $\mathcal{M}_a^{\lambda_i}(E)$  (see section 4.1). But such a quotient is a locally trivial  $K$ -bundle over a circle, where  $K$  is the cone over  $\mathbb{P}^1 \simeq S^2$ , so it has a natural manifold structure. Therefore, for  $g \in \text{Met}_{\text{wgood}}^r(E)$ ,  $\mathcal{M}_a^{\text{ASD}}(E)$  is a compact 4-manifold, which can be oriented as in classical Donaldson theory (see [DK] p. 283).

The signature of this 4-manifold will be an invariant  $\theta(M)$  of the base 4-manifold (endowed with the usual orientation data). Indeed, the main point here is that the set  $\text{Met}_{\text{vgood}}^r(E)$  is connected. For two choices  $g_0, g_1 \in \text{Met}_{\text{vgood}}^r(E)$ , consider a path  $\gamma : [0, 1] \rightarrow \text{Met}_{\text{vgood}}^r(E)$  connecting these metrics. A generic deformation (with fixed ends) of  $\gamma$  will define a cobordism between the moduli spaces associated with  $g_i$ . Note that *the cobordism constructed in this way is always trivial around the reductions*.

**Remark.** *In the example given in section 4.1, one has  $\mathcal{M}_a^{\text{ASD}}(E) \simeq S^4$ , so the  $\theta$ -invariant vanishes.*

This shows that, if non-trivial, this invariant can also be regarded as an obstruction to the representability of the elements of the orthonormal basis  $\{e_1, e_2\}$  by embedded spheres.

## 5. APPENDIX

**5.1. Metric-dependent spaces of selfdual and anti-selfdual forms.** Let  $M$  a compact oriented connected 4-manifold and  $E$  a vector bundle. For every Riemannian metric  $g$  on  $M$  one has two associated spaces of  $E$ -valued (anti)selfdual forms  $A_{\pm}^2_g(E)$ . It is convenient to complete the space  $\text{Met}$  of metrics with respect to the  $\mathcal{C}^r$  topology and the spaces  $A_{\pm}^2_g(E)$  with respect to a Sobolev norm  $L_k^2$  (where  $r \gg k$ ). In this way one gets Banach vector bundles  $[\mathcal{A}_{\pm}^2(E)]_k$  on the Banach manifold  $\text{Met}^r$  of  $\mathcal{C}^r$ -metrics.

One can trivialize globally these bundles in the following way. Fix a  $\mathcal{C}^r$ -metric  $g_0$ . The space  $\text{Met}^r$  can be identified with the space of positive  $g_0$ -symmetric automorphisms of the tangent bundle  $T_M$  via the diffeomorphism  $h \mapsto g_h := h^*(g_0)$ . We get homeomorphisms

$$\Gamma(\text{Sym}^+(T_M, g_0))^r \times [A_{\pm}^2_{g_0}(E)]_k \xrightarrow{\simeq} [\mathcal{A}_{\pm}^2(E)]_k$$

given by  $(h, \eta) \mapsto (g_h, h^*(\eta))$ . It is important to notice that homeomorphisms associated with different metrics  $g_0$  are pairwise differentiable compatible. Therefore one can use these homeomorphisms to define structures of Banach manifolds on the total spaces  $[\mathcal{A}_{\pm}^2(E)]_k$ .

A positive  $g_0$ -symmetric automorphism  $h \in \Gamma(\text{Sym}^+(T_M, g_0))^r$  defines a class  $\mathcal{C}^r$  positive  $g_0$ -symmetric automorphism  $\Lambda^2 h$  of the bundle  $\Lambda_M^2$ , given by  $\lambda \mapsto h^*(\lambda)$ . Using the  $g_0$ -orthogonal decomposition  $\Lambda_M^2 = \Lambda_{+g_0}^2 \oplus \Lambda_{-g_0}^2$ , the automorphism  $\Lambda^2 h$

can be written as

$$\Lambda^2 h = \begin{pmatrix} \mu_+^+(h) & \mu_+^-(h) \\ \mu_+^-(h) & \mu_+^+(h) \end{pmatrix},$$

where  $\mu_{\pm}^{\pm}(h)$  are symmetric endomorphisms of  $\Lambda_{\pm g_0}^2$  and  $\mu_{\mp}^{\pm}(h) : \Lambda_{\mp g_0}^2 \rightarrow \Lambda_{\pm g_0}^2$  have the property  $\mu_+^-(h)^* = \mu_+^-(h)$ . The tangent space of  $\Gamma(\text{Sym}^+(T_M, g_0))^r$  at  $\text{id}$  can be identified with  $A^0(\text{Sym}(T_M, g_0))^r$ . For a symmetric endomorphism  $\chi$  we put

$$m(\chi) := \left. \frac{\partial \mu(h)}{\partial h} \right|_{\text{id}}(\chi), \quad m_{\pm}^{\pm}(\chi) := \left. \frac{\partial \mu_{\pm}^{\pm}(h)}{\partial h} \right|_{\text{id}}(\chi), \quad m_{\mp}^{\pm}(\chi) := \left. \frac{\partial \mu_{\mp}^{\pm}(h)}{\partial h} \right|_{\text{id}}(\chi).$$

Since  $m(\chi)$  is symmetric, one has  $m_+^+(\chi)^* = m_+^-(\chi)$ .

**Lemma 5.1.** *The map  $\chi \mapsto m_{\mp}^{\pm}(\chi)$  defines an isomorphism*

$$A^0(\text{Sym}_0(T_M, g_0))^r \simeq A^0(\text{Hom}(\Lambda_{\mp}^2, \Lambda_{\pm}^2))^r,$$

where  $\text{Sym}_0(T_M, g_0)$  denotes the bundle of trace free  $g_0$ -symmetric endomorphisms of  $T_M$ .

This can be checked easily by diagonalizing  $\chi$  in an orthonormal frame.

The space  $A^0(\text{Sym}(T_M, g_0))^r$  can be regarded as the space of infinitesimal variations of  $\mathcal{C}^r$ -metrics, whereas  $A^0(\text{Sym}_0(T_M, g_0))^r$  is the space of infinitesimal variations of  $\mathcal{C}^r$ -conformal structures.

**5.2. Admissible metrics.** Consider an oriented compact connected manifold  $M$  and a cohomology class  $l \in H_{\text{DR}}^p(M)$ , where  $1 \leq p \leq n-1$ . For a Riemannian metric  $g$  on  $M$  we will denote by  $l_g$  the unique  $g$ -harmonic representative of  $l$ .

Let  $M$  a 4-manifold. The family of vector spaces

$$(\Lambda_{\pm g, x}^2)_{g \in \text{Met}^r, x \in M}$$

defines a bundle  $\Omega^{\pm}$  on the product space  $\text{Met}^r \times M$ . Suppose now that  $b_+(M) = 0$ . In this case any harmonic 2-form is ASD. For a de Rham 2-cohomology class  $l$ , the assignment  $\text{Met}^r \times M \ni (g, x) \mapsto l_g(x)$  defines a universal  $\mathcal{C}^{r-\epsilon}$  section  $\lambda$  in the bundle  $\Omega^-$  over  $\text{Met}^r \times M$ .

**Lemma 5.2.** *Suppose that  $b_+(M) = 0$  and  $l \in H_{\text{DR}}^2(M, \mathbb{R}) \setminus \{0\}$ . The associated universal section  $\lambda$  is submersive at every vanishing point.*

**Proof:** Let  $(g_0, x_0)$  be a vanishing point of  $\lambda$ . We use the metric  $g_0$  to simultaneously parameterize the space of metrics (using positive  $g_0$ -symmetric endomorphisms) and to trivialize the bundles  $\Omega^{\pm}$ . In this way, our section gives rise to a map

$$\Gamma(\text{Sym}^+(T_M, g_0))^r \times M \ni (h, x) \mapsto \omega_h(x) \in \Lambda_{-, x}^2.$$

where  $\omega_h := [\mu(h)^{-1}]^*(l_{g_h}) \in A_-^2(M)$ . Put  $\omega_0 := \omega_{\text{id}} = l_{g_0}$ . Writing  $l_{g_h} = \omega_0 + d\alpha_h$  (with  $\alpha_h \perp_{L^2} Z^1$ ), one gets for the derivative  $\dot{\alpha}(\chi) := \left. \frac{d\alpha}{dh} \right|_{\text{id}}(\chi)$  the equation

$$p^+ [-m(\chi)^*(\omega_0) + d\dot{\alpha}(\chi)] = 0,$$

which gives the solution

$$\dot{\alpha}(\chi) = G^+(m(\chi)_+^-(\omega_0)),$$



where  $G^+$  is the  $[Z^1]^\perp$ -valued inverse of the operators  $d^+$ . Here  $Z^1 \subset A^1(M)$  denotes as usually the space of closed 1-forms. Therefore

$$\left. \frac{d\omega_h}{dh} \right|_{\text{id}} (\chi) = dG^+(m_-^+(\chi)(\omega_0)) - m(\chi)(\omega_0) = d^-G^+(m_-^+(\chi)(\omega_0)) - m_-^-(\chi)(\omega_0) .$$

Therefore

$$\left. \frac{\partial \omega_h(x)}{\partial h} \right|_{(\text{id}, x_0)} (\chi) = d^-G^+(m_-^+(\chi)(\omega_0))(x_0) ,$$

because  $\omega_0(x_0) = 0$ . Suppose now that  $v \in [\Lambda_{x_0}^-]$  is orthogonal on the range of this partial derivative. The vector  $v$  defines a Dirac type ASD degree 2-current  $v_{x_0}$ . Since any element of  $m \in A^0(\text{Hom}(\Lambda_-^2, \Lambda_+^2))$  has the form  $m_-^+(\chi)$ , we obtain that for any such  $m$

$$(19) \quad 0 = \langle v, d^-G^+(m(\omega_0))(x_0) \rangle = \langle v_{x_0}, d^-G^+(m(\omega_0)) \rangle .$$

The assignment

$$\psi \mapsto \langle v_{x_0}, d^-G^+(\psi^+) \rangle$$

defines a self-dual degree 2-current, and the identity (19) implies that this current vanishes on  $M \setminus Z(\omega_0)$ . This current has the following important properties:

- (1) One has  $d(\delta) = d(v_{x_0})$ .
- (2)  $\delta$  is supported at  $x_0$ .

(1) Indeed, to prove the first claim, consider a test 1-form  $\alpha$ , and compute

$$\langle \delta, d\alpha \rangle = \langle v_{x_0}, d^-G^+d^+\alpha \rangle = \langle v_{x_0}, d^- \text{pr}_{Z^1}(\alpha) \rangle = \langle v_{x_0}, d^-\alpha \rangle = \langle v_{x_0}, d\alpha \rangle .$$

(2) Using the fact that  $d$  is overdetermined elliptic on  $A_+^2(M)$ , it follows from the first statement that  $\delta|_{M \setminus \{x_0\}}$  is smooth. On the other hand, by (19),  $\delta$  vanishes on the dense open set  $M \setminus Z(\omega_0)$ . Therefore  $\delta$  must vanish everywhere on  $M \setminus \{x_0\}$ .

The statement follows now from the following Lemma. ■

**Lemma 5.3.** *Let  $u$  be a  $\Lambda^2$ -valued distribution on  $\mathbb{R}^4$  which is supported at the origin and is closed. If  $u^-$  has order 0, then  $u = 0$ .*

**Proof:** If  $du = 0$  and  $u^-$  has order 0, then  $u^+$  must also have order 0, because, if not, taking the sum  $u_k^+$  of all terms of highest order  $k \geq 1$  in the decomposition of  $u^+$  as sum of partial derivatives of Dirac type distributions, one would have  $du_k^+ = 0$ . Therefore  $u_k^+$  would be singular harmonic self-dual form, which is impossible.

Therefore,  $u$  is an order zero Dirac type distribution, so it can be written as  $u = \sum_{i < j} a_{ij} \delta_0 e^i \wedge e^j$ , where  $\delta_0$  is the standard Dirac distribution. We get

$$du = \sum_{i < j < k} \left( a_{jk} \frac{\partial}{\partial x_i} |_0 - a_{ik} \frac{\partial}{\partial x_j} |_0 + a_{ij} \frac{\partial}{\partial x_k} |_0 \right) e^i \wedge e^j \wedge e^k ,$$

and the relation  $du = 0$  implies obviously  $a_{ij} = 0$  for all  $i < j$ . ■

**Definition 5.4.** *Let  $M$  be a 4-manifold with  $b_+(M) = 0$  and  $l \in H_{\text{DR}}^2(M) \setminus \{0\}$ . A metric  $g \in \text{Met}^r$  will be called*

- (1) *strictly  $l$ -admissible, if  $l_g$  is submersive (transversal to the zero section) at any vanishing point.*

- (2)  $l$ -admissible, if the intrinsic derivative of the section  $l_g \in \Gamma(\Lambda_{-,g}^2)$  at any vanishing point has rank at least 2.

Denote by  $\text{Met}_3^r(l)$  and  $\text{Met}_{\geq 2}^r(l)$  the space of (strictly) admissible  $\mathcal{C}^r$ -metrics.

- Proposition 5.5.** (1) *Suppose that the regularity class  $r$  is sufficiently large. The space  $\text{Met}_3^r$  of  $l$ -strictly admissible metrics is open and dense in  $\text{Met}^r$ .*  
 (2) *The space  $\text{Met}_{\geq 2}^r$  of  $l$ -admissible metrics is open, dense and path connected in  $\text{Met}^r$ .*

**Proof:** The openness of the two sets is obvious taking into account the compactness of the manifold, the continuity of the map  $g \mapsto l_g$  with respect to the  $\mathcal{C}^1$  topology on the space of sections, and the fact that, in general, for any bundle  $E$  and fixed base point  $x$ , the condition

$$s(x) = 0, \text{rk}(D_x(s)) \leq k$$

is closed with respect to the  $\mathcal{C}^1$ -topology on the space of sections in  $E$ .

To prove the density of the set of admissible metrics, note that the vanishing locus  $Z(\lambda)$  of the universal section  $\lambda$  is a smooth codimension 3-submanifold of  $\text{Met}^r \times M$ , and the natural map  $q : Z := Z(\lambda) \rightarrow \text{Met}^r$  is Fredholm of index 1.  $\text{Met}_3^r(l)$  is just the set of regular values of  $q$ .

Therefore the larger set  $\text{Met}_{\geq 2}^r(l)$  will also be dense. In order to prove the fact that this set is also path connected, consider – for two given metrics  $g_0, g_1 \in \text{Met}_{\geq 2}^r(l)$  – a differentiable path  $\gamma : [0, 1] \rightarrow \text{Met}^r$  joining them.

Consider that map

$$G : \Gamma(SL(T_X))^r \times [0, 1] \longrightarrow \text{Met}^r$$

defined by  $G(h, t) = h^*(\gamma(t))$ . Since the partial derivative  $\frac{\partial G}{\partial h}$  alone is surjective at any point, we conclude that the map  $G$  is transversal to  $q$ , so the fibered product  $P := \{(h, t, z) | G(h, t) = q(z)\}$  is a smooth submanifold of  $\Gamma(SL(T_X))^r \times [0, 1] \times Z$ . The natural map  $P \rightarrow \Gamma(SL(T_X))^r$  is proper and Fredholm of index 2. By Sard-Smale theorem [Sm], in any neighborhood of  $\text{id}$  one can find a regular value  $h_0$  of this map. Therefore, for such  $h_0$

$$P_0 := \{(t, z) | G(h_0, t, z) = q(z)\}$$

is a smooth 2-dimensional submanifold of  $[0, 1] \times Z$ . Put  $g'_t := G(h_0, t)$ . The fiber over  $t \in [0, 1]$  is contained in  $\{t\} \times \{g'_t\} \times M$  and is identified with the vanishing locus  $Z(l_{g'_t})$  of the section  $l_{g'_t}$  under the projection on the third factor.

We claim that the path  $G(h_0, \cdot)$  takes values in  $\text{Met}_{\geq 2}^r(l)$ . Indeed, the Zariski tangent space of  $Z(l_{g'_t})$  at a point  $x$  is just the intersection of the tangent space of  $P_0$  at  $(t, g'_t, x)$  with the tangent space of the fiber  $\{t\} \times \{g'_t\} \times M \subset P_0$  over  $t$ . But the tangent space of  $P_0$  at any point is 2-dimensional, so the Zariski tangent space of  $Z(l_{g'_t})$  at  $x$  has dimension 1 or 2.

In order to complete the proof, it suffices to join  $g_0$  to  $g'_0$  and  $g_1$  to  $g'_1$  with paths in  $\text{Met}_{\geq 2}^r(l)$ . By the openness property of this space, it follows easily that (if  $h_0$  is sufficiently close to  $\text{id}$ ) the metrics  $G(\text{id} + s(h_0 - \text{id}), 0)$ ,  $G(\text{id} + s(h_0 - \text{id}), 1)$ ,  $s \in [0, 1]$  remain in  $\text{Met}_{\geq 2}^r(l)$  for every  $s \in [0, 1]$ .  $\blacksquare$

Note that we see no reason why the space  $\text{Met}_3^r(l)$  should be connected. Indeed, suppose that for two metrics  $g_0, g_1 \in \text{Met}_3^r(l)$ , the corresponding vanishing loci (which are finite unions of pairwise disjoint embedded circles) have different number

of connected components. Then there is certainly no way to join the two metrics by a path in  $\text{Met}_3^r(l)$ .

**Remark.** *If  $g \in \text{Met}_{\geq 2}^r(l)$ , then any point  $x \in Z(l_g)$  has a neighborhood  $U_x \subset M$  such that  $Z(l_g) \cap U$  is contained in a closed submanifold  $N_x$  of  $U_x$  of dimension 1 or 2. In particular the vanishing locus  $Z(l_g)$  of the  $g$ -harmonic ASD form  $l_g$  has Hausdorff dimension  $\leq 2$ .*

**Proof:** Indeed, consider a smooth map  $\mathbb{R}^n \supset V \xrightarrow{f} \mathbb{R}^m$  whose rank at  $0 \in V$  is  $k$  and let  $E := \text{im}(d_0(f))$ . Then the composition  $p_E \circ f$  is a submersion at 0, so its restriction on a sufficiently small neighborhood  $U$  of 0 will be a submersion. Note that  $Z(f) \subset Z(p_E \circ f)$  and that  $Z(p_E \circ f) \cap U = Z((p_E \circ f)|_U)$  is a codimension  $k$  submanifold of  $U$ .  $\blacksquare$

The results above can be extended for twisted de Rham cohomology classes: Let  $\rho : \pi_1(M) \rightarrow \mathbb{Z}_2$  be an epimorphism, and suppose that  $b_+(M_\rho) = 0$  where  $\pi_\rho : M_\rho \rightarrow M$  is the corresponding double cover of  $M$ . Let  $l \in H_{\text{DR}}^2(M_\rho) \setminus \{0\}$  be a de Rham cohomology class with the property that  $\iota^*(l) = -l$ , where  $\iota$  stands for the tautological involution of  $M_\rho$ . Then, for every metric  $g$  on  $M$ , the  $\pi_\rho^*(g)$ -harmonic representative  $l_g$  of  $l$  is an ASD form on  $M_\rho$  satisfying the identity  $\iota^*(l_g) = -l_g$ . In other words,  $l_g$  is a  $\rho$ -twisted ASD form on  $M$ . At any point  $x \in M$ ,  $l_g$  is defined up to sign. Therefore, the vanishing locus  $Z(l_g) \subset M$  and the rank of the intrinsic derivative at a vanishing point are well-defined objects. In particular one can associate to  $l$  the sets of metrics  $\text{Met}_3^r(l)$ ,  $\text{Met}_{\geq 2}^r(l)$  as in the non-twisted case, and these sets also have the properties stated in Proposition 5.5.

**Proposition 5.6.** *Let  $M$  be a 4-manifold and  $d \in H^2(M, \mathbb{Z})$*

(1) *Suppose that*

$$(20) \quad b_+(M) = 0 \text{ and } d \notin 2H^2(M, \mathbb{Z}) + \text{Tors} .$$

*Then, for every  $c \in H^4(M, \mathbb{Z})$  the set*

$$\text{Met}_{\text{adm}}^r(c) := \bigcap_{\substack{l \in H^2(M, \mathbb{Z}) \\ \iota \cdot (d-l) \leq c}} \text{Met}_{\geq 2}^r(2l - d)$$

*is open, dense and connected in  $\text{Met}^r$ .*

(2) *Suppose that (20) holds and, for every epimorphism  $\rho : \pi_1(M, x_0) \rightarrow \mathbb{Z}_2$*

$$(21) \quad b_+(M_\rho) = 0 \text{ and } \pi_\rho^*(d) \notin 2H^2(M_\rho, \mathbb{Z}) + \text{Tors} .$$

*Then the set*

$$\text{Met}_{\text{tadm}}^r(c) := \text{Met}_{\text{adm}}^r(c) \bigcap \left[ \bigcap_{\substack{\pi_1(M, x_0) \xrightarrow{\rho} \mathbb{Z}_2, l \in H^2(M_\rho, \mathbb{Z}) \\ l + \iota^*(l) = \pi_\rho^*(d), \iota \cdot \iota^*(l) \leq \pi_\rho^*(c)}} \text{Met}_{\geq 2}^r(l - \iota^*(l)) \right]$$

*is open, dense and connected in  $\text{Met}^r$ .*

**Proof:** The important point here is that, since the intersection form of  $M$  is negative definite, the set of terms in the first intersection is finite. For the second intersection note first that  $\rho$  varies in a finite set (isomorphic to the set of index 2

subgroups of  $H_1(M, \mathbb{Z})$  and, under our assumption, for any fixed  $\rho$ , there are only finitely many possibilities for  $l$ . To complete the proofs it suffices to note that the two properties in Proposition 5.5 have been obtained by applying the Sard-Smale theorem to certain proper Fredholm maps, and using the fact that the set of regular values of such a map is open and dense. A finite intersection of such sets will also be open and dense.  $\blacksquare$

**5.3. Analytic results.** We begin with the following easy result concerning the range of a proper Fredholm map  $f : V \rightarrow W$  of negative index  $j$ .

**Lemma 5.7.** *Let  $V, W$  be separable Banach manifolds and  $f : V \rightarrow W$  a proper, smooth, Fredholm map of negative index  $j \leq -1$ . Then*

- (1) *The range  $f(V)$  of  $f$  is closed and nowhere dense,*
- (2) *When  $j \leq -2$  the natural map  $\pi_i(W \setminus f(V)) \rightarrow \pi_i(W)$  is an isomorphism for  $0 \leq i \leq -j - 2$  and is surjective for  $i = -j - 1$ .*

Note that we do not assume that  $W$  is connected. The second statement claims that  $W_0 \setminus f(V)$  is connected for every connected component  $W_0$  of  $W$ , and that the map  $\pi_i(W_0 \setminus f(V), x_0) \rightarrow \pi_i(W_0, x_0)$  is an isomorphism (respectively surjective) for every  $x_0 \in W_0 \setminus f(V)$  and  $0 \leq i \leq -j - 2$  ( $i = -j - 1$ ).

**Proof:** The first statement follows directly from the Sard-Smale theorem: Since  $\text{ind}(f) < 0$  a point  $w \in W$  is a regular value for  $f$  if and only if  $w \notin f(V)$ . The second statement is proved using standard transversality arguments (see [DK] section 4.3.2). We illustrate this method in the case  $i = 0$ . For this case we have to prove that  $\pi_0(W \setminus f(V)) \rightarrow \pi_0(W)$  is injective (the surjectivity follows from (1)). Consider two points  $w_0, w_1 \in W \setminus f(V)$  belonging to the same component  $W_0$  of  $W$  and connect them with a smooth path  $\gamma : [0, 1] \rightarrow W_0$ . Using Proposition 4.3.10 [DK] we obtain a smooth path  $\gamma' : [0, 1] \rightarrow W_0$  with  $\gamma'|_{\{0,1\}} = \gamma|_{\{0,1\}}$  arbitrary close to  $\gamma$  which is transversal to  $f$ . Since  $\text{ind}(f) \leq -2$ , this transversality condition implies  $\text{im}(\gamma') \cap f(V) = \emptyset$ .  $\blacksquare$

The following proposition plays an important role in the proof of our regularity results. This will allow us to refine the Freed-Uhlenbeck theorems [FU] and to estimate the codimension of the spaces of bad metrics.

**Proposition 5.8.** *Let  $M$  be an  $n$ -dimensional compact manifold,  $E$  a real rank  $r$  vector bundle and  $\nabla$  a linear connection on  $E$ . Let  $U \subset M$  be an open set whose complement  $\Sigma$  has Hausdorff dimension  $d \leq n - 2$ . Let  $\alpha \in A^1(E)$  and  $\varphi \in \Gamma(U, E)$  such that  $\nabla\varphi = \alpha|_U$ . Then  $\varphi$  extends smoothly to a section  $\psi \in \Gamma(M, E)$  satisfying  $\nabla\psi = \alpha$ .*

The idea of the proof is to use the (classical) methods of characteristics to solve the first order equation  $\nabla f = \alpha$ : we integrate the family of ordinary equations obtained by restricting our equation to an  $(n - 1)$ -dimensional family of embedded paths which define a local foliation. We choose the paths such that all starting points belong to  $U$ , and we use the values of  $\varphi$  at these points as initial conditions. The condition on the Hausdorff dimension of  $\Sigma$  implies that a dense family of paths do not meet the set  $\Sigma$  (where  $\varphi$  is not defined), so on these paths the sections  $f$  and  $\varphi$  coincide. In this way one checks that the section  $f$  obtained by pathwise integration agrees with  $\varphi$  on the intersections of their domains, providing a proper

extension of  $\varphi$ .

**Proof:** For every smooth path  $\gamma : (-1, 1) \rightarrow M$ , consider the connection  $\gamma^*(\nabla)$  on the bundle  $\gamma^*(E)$  on  $(-1, 1)$  and the affine ordinary differential equation

$$(22) \quad \gamma^*(\nabla)g = \gamma^*(\alpha)$$

for sections  $g$  in the bundle  $\gamma^*(E)$ . Using the general theory of ordinary differential equations, one gets, for every  $e \in E_{\gamma(0)}$ , a unique solution

$$g_{\gamma,e} \in \Gamma((-1, 1), \gamma^*(E))$$

of the equation (22) satisfying the initial condition  $g_{\gamma,e}(0) = e$ . This solution depends differentiably on the pair  $(\gamma, e)$ , where  $e \in E_{\gamma(0)}$ . Our hypothesis  $\nabla\varphi = \alpha|_U$  implies  $\varphi \circ \gamma = g_{\gamma,\varphi(\gamma(0))}$  for every smooth path  $\gamma : (-1, 1) \rightarrow U$ .

Let  $\psi \in \Gamma(V, E)$  be a maximal element of the ordered set of extensions of  $\varphi$  defined on open subsets of  $M$ . The existence of such a maximal element follows by the Zorn lemma. Since  $U$  is dense in  $M$  (so also in  $V$ ) one has

$$(23) \quad \nabla\psi = \alpha|_V$$

which implies

$$(24) \quad \psi \circ \gamma = g_{\gamma,\psi(\gamma(0))}$$

for every smooth path  $\gamma : (-1, 1) \rightarrow V$ . The complement  $\Sigma' = M \setminus V$  will also have Hausdorff dimension  $\leq n - 2$ .

We claim that  $V = M$ . Suppose not, and let  $x_0 \in M \setminus V$ . Denoting by  $B^{n-1}$  the standard ball in  $\mathbb{R}^{n-1}$  consider a local parametrization

$$h : B^{n-1} \times (-1, 1) \xrightarrow{\cong} W \subset M$$

of  $M$  such that

$$h(B^{n-1} \times \{0\}) \subset V, \quad h(0, \frac{1}{2}) = x_0.$$

The idea is to extend  $\psi$  on  $V \cup W$  using the solutions  $g_{h_x, \psi(h(x,0))}$ ,  $x \in B^{n-1}$ , where  $h_x$  denotes the path  $t \mapsto h(x, t)$ .

So put

$$f(x, t) = g_{h_x, \psi(h(x,0))}(t), \quad \eta := f \circ h^{-1}.$$

We claim that  $\eta|_{W \cap V} = \psi|_{W \cap V}$ , which will complete the proof, because this would yield a proper extension of  $\psi$ , contradicting its maximality.

Our claim is equivalent to

$$f|_{h^{-1}(V)} = \psi \circ h|_{h^{-1}(V)}.$$

The two functions coincide on  $[B^{n-1} \setminus \text{pr}_1(h^{-1}(\Sigma'))] \times (-1, 1)$ , because, for any  $x \in [B^{n-1} \setminus \text{pr}_1(h^{-1}(\Sigma'))]$  the corresponding path  $h_x$  is entirely contained in  $V$  so, for such  $x$ , both sections

$$t \mapsto f(x, t), \quad t \mapsto \psi(h(x, t))$$

coincide with  $g_{h_x, \psi(h(x,0))}$ , by (24) and the definition of  $f$ . It suffices to notice that, by our hypothesis, the set  $h^{-1}(\Sigma')$  is of Hausdorff dimension at most  $n - 2$ , so its projection on the  $(n - 1)$  dimensional ball  $B^{n-1}$  is also of Hausdorff dimension at most  $n - 2$ . Therefore, this projection cannot contain any non-empty open set, so its complement in  $B^{n-1}$  is dense, so  $[B^{n-1} \setminus \text{pr}_1(h^{-1}(\Sigma'))] \times (-1, 1)$  is dense in  $B^{n-1} \times (-1, 1)$ , so the two functions coincide everywhere.  $\blacksquare$

The following corollary shows that the statement of Lemma 4.16 in [FU] is true as soon as the vanishing of the ASD curvature has Hausdorff dimension  $\leq 2$ . In particular this statement is true for  $C^\infty$ -metrics by the results of [B] and for admissible metrics.

**Corollary 5.9.** *Let  $(M, g)$  be a compact oriented Riemannian 4-manifold endowed with a  $C^r$ -metric  $g$ , and  $S$  a Hermitian line bundle on  $M$  endowed with a non-flat ASD connection  $\sigma$ . Suppose that the vanishing of the curvature has Hausdorff dimension  $\leq 2$ . Let  $\beta \in A^1(S)$  such that*

- (1)  $d_\sigma^* \beta = 0, d_\sigma^+ \beta = 0,$
- (2) *The anti-selfdual  $S$ -valued form  $d_\sigma \beta$  is a tensor multiple of  $F_\sigma$  at any point  $x \in M$  for which  $F_{\sigma, x} \neq 0$ .*

Then  $\beta = 0$ .

**Proof:** Let  $U$  be the complement of the vanishing locus  $\Sigma$  of  $F_\sigma$ . By assumption we can write  $d_\sigma \beta|_U = F_\sigma \otimes \zeta$ , for a section  $\zeta \in \Gamma(U, S)$ . We get

$$[F_\sigma|_U] \wedge (\beta|_U - d_\sigma \zeta) = d_\sigma d_\sigma (\beta|_U - d_\sigma \zeta) = d_\sigma (d_\sigma \beta|_U - F_\sigma \otimes \zeta) = 0,$$

hence  $\beta|_U - d_\sigma \zeta = 0$ , since the wedge product with a non-trivial (real or purely imaginary) ASD form is invertible on 1-forms. By Proposition 5.8 and the assumption on  $Z(F_\sigma)$ , the section  $\zeta$  extends smoothly to a section  $\xi$  on  $M$  satisfying  $d_\sigma \xi = \beta$ . Since we assumed  $d_\sigma^* \beta = 0$ , we get immediately  $\beta = 0$ . ■

**Proposition 5.10.** *Let  $S$  be a Hermitian line bundle on an oriented Riemannian 4-manifold  $(M, g)$  and  $\sigma$  a Hermitian connection on  $S$ . Let  $\eta \in A_+^2(S)$  with  $d_\sigma \eta = 0$ . Suppose that on an open set  $U \subset M$ , the form  $\eta$  (regarded as section in  $\Lambda_+^2 \otimes S$ ) has real rank 1. Then  $F_\sigma|_U = 0$ .*

**Proof:** Supposing that  $U$  is simply connected, we can write  $\eta = \omega \otimes \zeta$ , where  $\omega \in A_+^2(U)$  is a real selfdual form, and  $\zeta \in \Gamma(U, S)$ . By assumption, both  $\omega$  and  $\zeta$  are nowhere vanishing on  $U$ . Since  $d_\sigma \eta = 0$ , we get

$$d\omega \otimes \zeta + \omega \wedge d_\sigma \zeta = 0,$$

hence  $\omega \wedge (\theta \otimes \zeta + d_\sigma \zeta) = 0$ , where  $\theta$  is the real form on  $U$  defined by  $d\omega = \omega \wedge \theta$ . We get

$$d_\sigma \zeta = -\theta \otimes \zeta, \quad F_\sigma \otimes \zeta = -d\theta \otimes \zeta + \theta \wedge d_\sigma \zeta = -d\theta \otimes \zeta - (\theta \wedge \theta) \otimes \zeta = -d\theta \otimes \zeta.$$

This yields  $F_\sigma = -d\theta$ , in which the left hand term is purely imaginary and the right hand term is real. ■

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